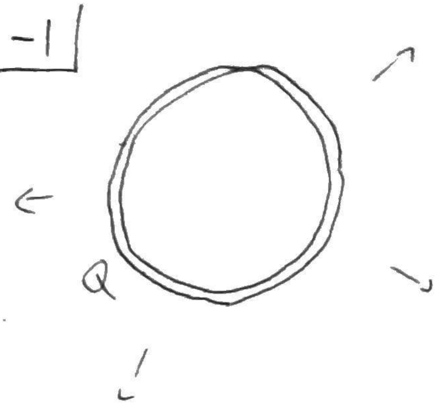
8 - Gauss' Law

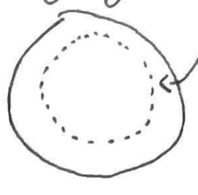
29-1



(a) $\vec{E} = 0$ always,

since inside uniform spherical shell.

(By gauss' law



$Q_{enc} = 0 \Rightarrow \Phi = 0$

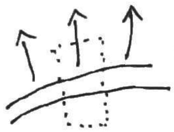
$\Rightarrow \vec{E} = 0$

+ spherical sym.)

(b) As the balloon radius R increases, $SA = 4\pi R^2$,

and $Q = $ const $\Rightarrow \sigma = Q/SA = \frac{Q}{4\pi R^2}$

The surface charge density falls like $1/R^2$.



Near the ext. surface, $\vec{E} \cong \frac{\sigma}{\varepsilon_0} \hat{r}$

~~So/A~~ So the surface field strength ~~falls off~~ decreases like $1/R^2$

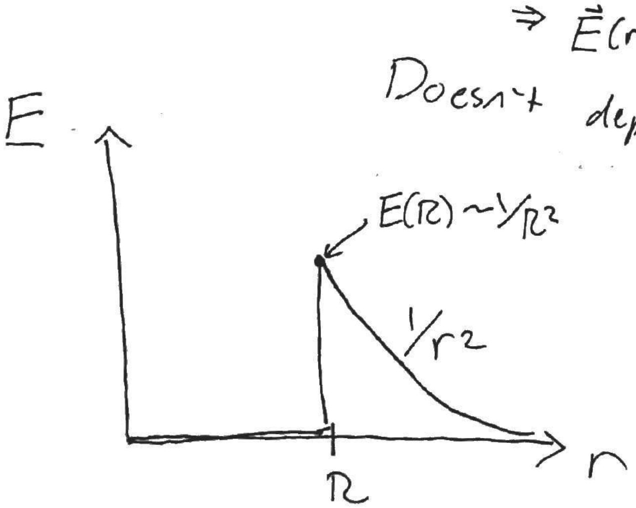
(c) Outside the balloon:

$Q_{enc} = Q, \quad A_{Gauss} = 4\pi r^2$ ~~$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}$~~

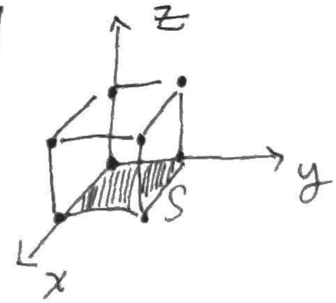
$\Phi = Q/\varepsilon_0 = \int \vec{E} \cdot d\vec{A} = E(r) \cdot 4\pi r^2$

$\Rightarrow \vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}$, just like pt. charge.

Doesn't depend on R.



29-2



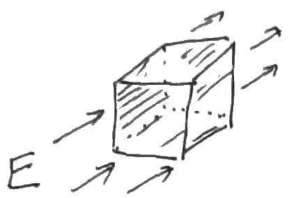
$$\Phi = \int_S \vec{E} \cdot d\vec{A} = \int_0^L dx \int_0^L dy \; \vec{E} \cdot \hat{k}$$

$$= \int_0^L dx \int_0^L dy \; E_z$$

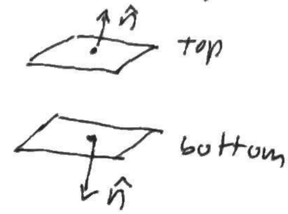
Uniform $E \Rightarrow = L^2 \cdot E_z$

(a) $\Phi = L^2 \cdot 6$ N/C (b) $\Phi = 0$ (c) $\Phi = 4L^2$ N/C

(d) Now we consider cube as <u>closed</u> surface:



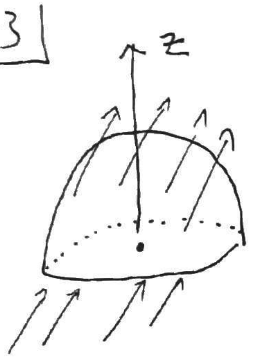
$$\left.\begin{aligned} &\hookrightarrow \Phi_{top} = -\Phi_{bot} \\ &\Phi_{left} = -\Phi_{right} \\ &\Phi_{front} = -\Phi_{back} \end{aligned}\right\} \text{Because } \vec{E} \text{ uniform}$$



$$\Rightarrow \Phi = (\Phi_{top} + \Phi_{bot}) + (\Phi_{left} + \Phi_{right}) + (\Phi_{front} + \Phi_{back})$$

$$= 0$$

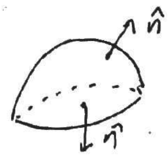
29-3



For an external uniform field like this,

$$\Phi_{\substack{\text{closed} \\ \text{surface}}} = Q_{enc} = 0$$

Take



as <u>closed surface</u>

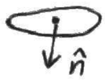
$$\Rightarrow \left.\begin{aligned} \Phi_{bottom} &= \int \vec{E} \cdot \\ \Phi_{top} &= \int \vec{E} \cdot \end{aligned}\right\} \quad \Phi_{top} + \Phi_{bot} = 0$$

$$\hookrightarrow \boxed{\Phi_{top} = -\Phi_{bot}}$$

(The − sign is b/c of the orientation for the bottom cap when we consider it as part of the closed surface. But we are free to reverse the sign of $\hat{n}$ when considering piece on its own.)

29-31 (Continued)

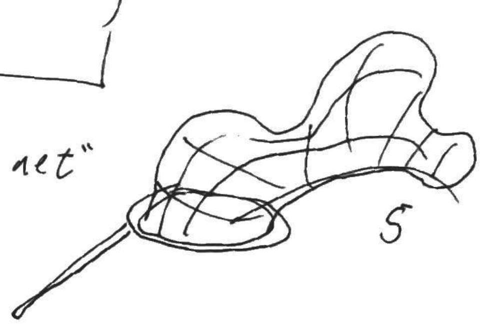
$$\Phi_{top} = \int \vec{E}\cdot\hat{n}\, dA = -\int \vec{E}\cdot\hat{n}\, dA = -\int \vec{E}\cdot(-\hat{z})\, dA = \int_{bot} E_z\, dA$$



※ Basically, however much flux goes into the ~~bottom~~ circle, base, comes out of the dome.

$$\boxed{\Phi_{Top} = \int_{base} E_z\, dA \underset{\text{uniform field}}{=} E_z \cdot \pi R^2}$$

FYI, this is true also for a "butterfly net" where the top is irregular:



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29-6

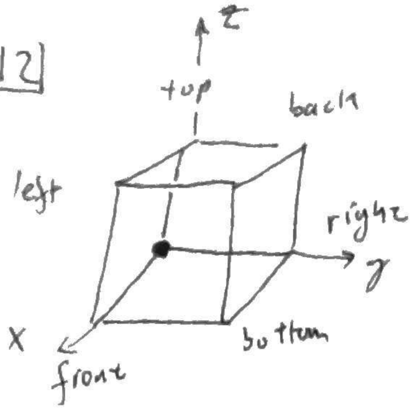
$$\Phi_{in} = 1 + 3 + 5 = 9$$

$$\Phi_{out} = 2 + 4 + 6 = 12$$

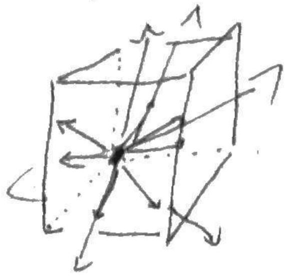
$$\Phi_{net} = \Phi_{out} - \Phi_{in} = 12 - 9 = 3 \cdot 10^3\ N\,m^2/C$$

$$\Rightarrow Q_{enc} \underset{G.L.}{=} \varepsilon_0 \Phi_{net} = \varepsilon_0 \cdot 3 \cdot 10^3\ N\,m^2/C$$

$\boxed{29-12}$



Sketching the $\vec{E}$ field from this charge:



~~E// sector~~

$\hat{E} \cdot \hat{n} = 0$ for the bottom, back, left sides:

The field points radially away from the charge, and flat along ~~surface~~ these faces $\Rightarrow \vec{E} \cdot \hat{n} = 0$.

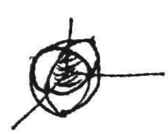
So $\Phi_{left} = \Phi_{bot} = \Phi_{back} = 0$

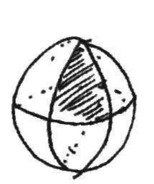
By symmetry (rotating coord. axes in particular $x \to y \to z$),

$$\Phi_{top} = \Phi_{right} = \Phi_{front}$$

How much of the pt charge is "enclosed"?

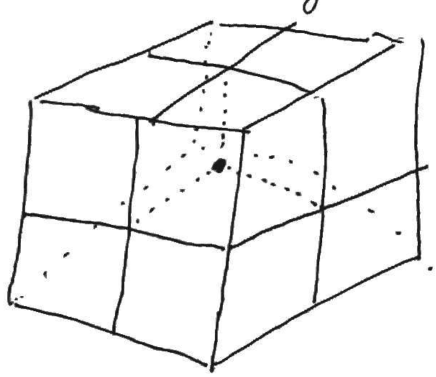
Imagine it as a small sphere: centered at origin





$\boxed{\text{1/8 of the sphere is included}}$

Can also imagine replicating the cube:



Then the pt charge lies at the center of this assembly of the 8 cubes.

Each cube should get an equal fraction ($\frac{1}{8}$) of the $\vec{E}$ flux, and the Q_{enc}.

29-12 (Continued)

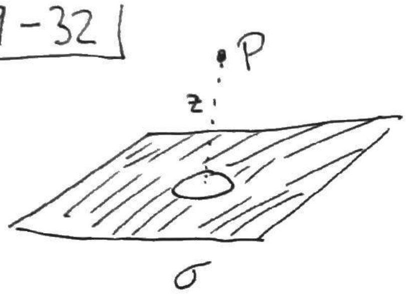
To summarize: $\boxed{\Phi_{left} = \Phi_{bot} = \Phi_{back} = 0}$

$$\Phi_{top} = \Phi_{right} = \Phi_{front} = \frac{1}{3}\Phi_{total}$$

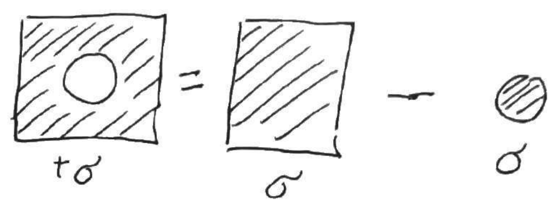
$$\Phi_{total} = Q_{enc}/\varepsilon_0 = \frac{q}{8\varepsilon_0}$$

$$\Rightarrow \boxed{\Phi_{top} = \Phi_{right} = \Phi_{front} = \frac{q}{24\varepsilon_0}}$$

29-32



The main thing is to ~~identi~~ use superposition

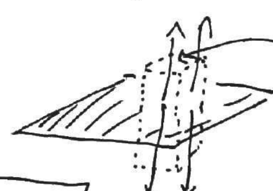


We consider the sheet-with-hole as a ~~superp~~ sum of:

- Uniform filled sheet $(+\sigma)$
- Uniform circular disk $(-\sigma)$

$$\Rightarrow \vec{E} = \vec{E}_{sheet} + \vec{E}_{disk}$$

For $\vec{E}_{sheet}$, use Gauss' law:



Prism w/ base area A.

$\Rightarrow Q_{enc} = \sigma A$

$\Phi = \Phi_{top} + \Phi_{bot}$ (rest are zero, $\hat{E} = \pm\hat{z}$)

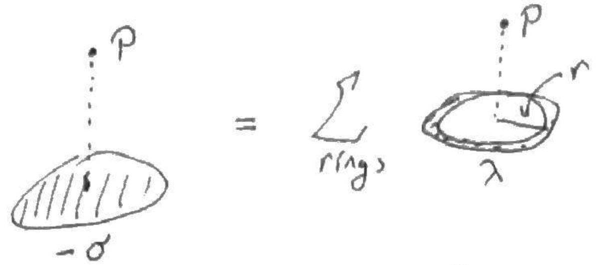
$= 2 \cdot E \cdot A$

$\Rightarrow E = \Phi/2A = \frac{Q_{enc}}{2\varepsilon_0 A} = \frac{\sigma A}{2\varepsilon_0 A}$

$$\Rightarrow \boxed{\vec{E}_{Sheet} = \frac{\sigma}{2\varepsilon_0}\hat{z}}$$

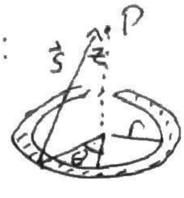
29-32 (continued)

For $\vec{E}_{disk}$, use a brute-force superposition integral.



Split disk into superposition of thin rings of radius r, thickness dr, Linear charge density $\lambda = -\sigma\, dr$

Ring:



$$dq = \lambda\, dl = \lambda\, r\, d\theta$$

$$dE_z = \frac{dq}{4\pi\varepsilon_0} \cdot \underbrace{\frac{1}{r^2+z^2}}_{\text{distance}^2 \text{ to P}} \cdot \underbrace{\frac{z}{(r^2+z^2)^{1/2}}}_{\text{unit vector projection to z}}$$

$$|s|^2 = r^2 + z^2$$

$$s_z = \frac{z}{(r^2+z^2)^{1/2}}$$

$$E_z = \int dE_z = \int_0^{2\pi} \frac{\lambda\, r\, d\theta}{4\pi\varepsilon_0} \frac{z}{(r^2+z^2)^{3/2}}$$

$$= \frac{\lambda z}{2\varepsilon_0} \frac{r}{(r^2+z^2)^{3/2}} = \frac{-\sigma z}{2\varepsilon_0} \frac{r\, dr}{(r^2+z^2)^{3/2}}$$

$E_x = E_y = 0$ by symmetry (reflection)

$$\Rightarrow E_z^{disc} = \int_0^R \frac{-\sigma z}{2\varepsilon_0} \frac{r\, dr}{(r^2+z^2)^{3/2}} = \frac{-\sigma z}{2\varepsilon_0} \int_0^R \frac{r\, dr}{(r^2+z^2)^{3/2}}$$

u-sub: $u = r^2 + z^2$, $du = 2r\, dr \Rightarrow$ $r = 0 \rightarrow u = z^2$, $r = R \rightarrow u = R^2 + z^2$

$$E_z^{disc} = \frac{-\sigma z}{4\varepsilon_0} \int_{z^2}^{R^2+z^2} \frac{du}{u^{3/2}} = \frac{\sigma z}{4\varepsilon_0}\left(-2\left[u^{-1/2}\right]_{z^2}^{R^2+z^2}\right)$$

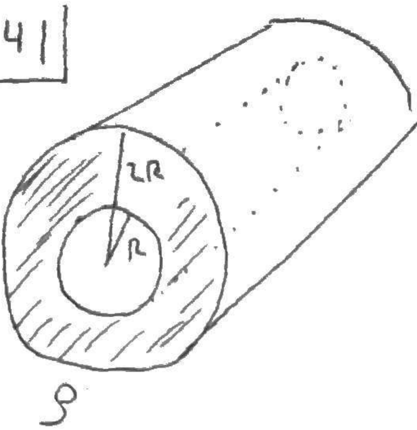
$$= \frac{-\sigma z}{2\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]$$

$$\boxed{E_z^{disc} = \frac{-\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]}, \quad E_x = E_y = 0$$

$$\Rightarrow \vec{E} = \vec{E}_{sheet} + \vec{E}_{disc} = \frac{\sigma}{2\varepsilon_0}\hat{z} - \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z} = \boxed{\frac{\sigma}{2\varepsilon_0} \frac{z}{\sqrt{R^2+z^2}}\hat{z}}$$

(Note for $z \gg R$, $\vec{E} \sim \frac{\sigma}{2\varepsilon_0}\hat{z}$, i.e. we can't tell the hole is there)

29-41



Take a cylindrical gaussian surface $C(r)$ with radius r, centered along the same axis as the charged shell.

length L

Cylindrical symmetry (rot. about axis, translation, reflections)

$$\Rightarrow \vec{E} = E(r)\,\hat{r} \,, \quad \hat{r} \text{ points away from axis.}$$

$$\Rightarrow \Phi(r) = \int_{C(r)} \vec{E}\cdot\hat{n}\, dA = \int_{C(r)} \vec{E}\cdot\hat{n}\, dA$$

$$= \int_{C(r)} E(r)\, dA = E(r)\cdot \int_{C(r)} dA = E(r)\left[2\pi r L\right]$$

$$\Rightarrow E(r) = \frac{\Phi(r)}{2\pi r L} \overset{G.L.}{=} \frac{Q_{enc}(r)}{2\pi\varepsilon_0 r L}$$

$$Q_{enc}(r) = \begin{cases} 0 & r < R \\ \rho\cdot L\cdot(\pi r^2 - \pi R^2) & R < r < 2R \\ \rho\cdot L\cdot(\pi(2R)^2 - \pi R^2) & 2R < r \\ \quad = 3\rho L \pi R^2 & \end{cases}$$

$$\Rightarrow E(r) = \begin{cases} 0 \\ \frac{\rho}{2\varepsilon_0}(r^2 - R^2)/r \\ \frac{3\rho}{2\varepsilon_0} R^2/r \end{cases} = \begin{cases} 0 & r < R \\ \rho/2\varepsilon_0\,(r - R^2/r) & R < r < 2R \\ 3\rho/2\varepsilon_0\; R^2/r & 2R \leq r \end{cases}$$

For $r = 2R$, $E(r) = \frac{3\rho}{2\varepsilon_0}\frac{R^2}{2R} = \frac{3\rho}{4\varepsilon_0} R$

~~This is halved when:~~ $\cancel{\frac{\rho}{2\varepsilon_0}\left(r - R^2/r\right) = \frac{1}{2}\,\frac{3\rho}{2\varepsilon_0} R}$

$\cancel{r - R^2/r = 3/2\, R}$

$\cancel{r^2 - \frac{3}{2} R r - R^2 = 0}$

$\cancel{\rightarrow r = \frac{1}{2}\left(\frac{3}{2}R \pm \sqrt{\frac{9}{4}R^2 + R^2}\right)}$

$\cancel{= \frac{1}{2}\left(\frac{3}{2}R \pm \sqrt{13}\, R/2\right)}$

$\cancel{= \frac{R}{4}\left(3 + \sqrt{13}\right)}$

$\cancel{\boxed{r = 1.65\, R}}$

29-41 (Continued) The field is at half its surface value when

$$E(r) = \frac{\rho}{2\varepsilon_0}\left(r - R^2/r\right) = \frac{1}{2}\cdot\frac{3\rho}{4\varepsilon_0}R$$

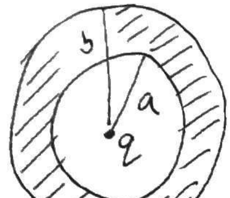
$$r - R^2/r = \frac{1}{2}\cdot\frac{3}{2}R$$

$$r^2 - \frac{3}{4}rR - R^2 = 0$$

$$\Rightarrow r = \frac{1}{2}\left(\frac{3}{4}R \pm \sqrt{\tfrac{9}{16}R^2 + 4R^2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{4}R \pm \frac{1}{4}\sqrt{73}\,R\right) = \boxed{\frac{1}{8}\left(3+\sqrt{73}\right)R} = 1.443\,R$$

29-42



$\rho = A/r$

Spherical symmetry $\Rightarrow \vec{E} = E(r)\hat{r}$

Take a spherical gaussian surface S_r w/ radius r, centered at origin.

$$\Phi(r) = \int_{S(r)} \vec{E}\cdot\hat{r}\,dA = \int_{S(r)} E(r)\underbrace{(\hat{r}\cdot\hat{r})}_{1}\,dA$$

$$= E(r)\int_{S(r)} dA = E(r)\cdot 4\pi r^2$$

$$\Rightarrow E(r) = \frac{\Phi(r)}{4\pi r^2} \overset{G.L.}{=} \frac{Q_{enc}}{4\pi\varepsilon_0}\frac{1}{r^2}$$

$r<a$: $Q_{enc} = q$, $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$

$a<r<b$:

$$Q_{enc} = q + \int_a^r\int_{\theta,\phi}\rho\,dV$$

$$= q + \int_a^r \frac{A}{s}\left(4\pi s^2\,ds\right)$$

$$= q + 4\pi A\int_a^r s\,ds$$

$$= q + 2\pi A\left(r^2 - a^2\right)$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q - 2\pi A a^2}{r^2} + 2\pi A\right)\hat{r}$$

$a<r<b$: $Q_{enc} = q + \int_a^b \rho\,dV = q + 2\pi A\left(b^2 - a^2\right)$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q + 2\pi A\left(b^2 - a^2\right)}{r^2}\hat{r}$$

29-42 (Continued)

In region $a<r<b$, $E(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q - 2\pi a^2 A}{r^2} + 2\pi A\right)$

Would have ~~uniform~~ const. magnitude if the $1/r^2$ term was 0.

$\Rightarrow \quad 0 = q - 2\pi a^2 A \Rightarrow \boxed{q = 2\pi a^2 A}$

(Note that this is $\int_0^a \rho\, dV$ - basically, we have taken all the A/r charge dist. in the $0<r<a$ region and squished it to the center.

)

29-44

(a) Spherical symmetry $\Rightarrow \vec{E} = E(r)\,\hat{r}$

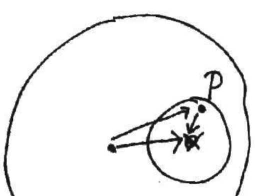
Take spherical gaussian surface S_r of radius r

$\Rightarrow \Phi(r) = \int_{S(r)} \vec{E}\cdot\hat{r}\, dA = E(r)\int_{S(r)} dA = 4\pi r^2 E(r)$

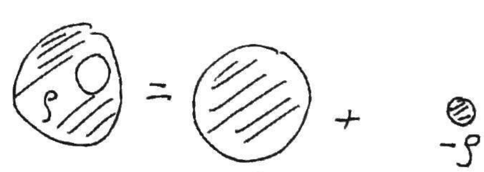
$Q_{enc} = \frac{4}{3}\pi r^3 \rho$

$\Rightarrow E = \Phi(r)/4\pi r^2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2} = \frac{r\rho}{3\varepsilon_0} \Rightarrow \boxed{\vec{E} = \frac{r\rho}{3\varepsilon_0}\hat{r} = \frac{\rho}{3\varepsilon_0}\vec{r}}$

(b)



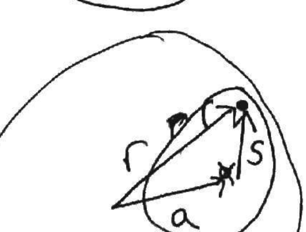
Superposition:

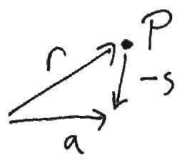


So for any P in the cavity, ← (in both the $+\rho$ and $-\rho$ uniform charges)

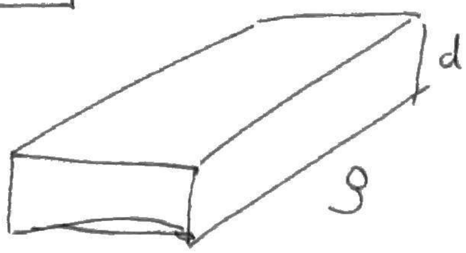
$\vec{E} = \vec{E}_{+\rho,\text{origin}} + \vec{E}_{-\rho,\vec{a}}$

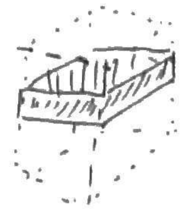
$= \frac{\rho}{3\varepsilon_0}\vec{r} - \frac{\rho}{3\varepsilon_0}\vec{s} = \frac{\rho}{3\varepsilon_0}(\vec{r} - \vec{s}) = \boxed{\frac{\rho}{3\varepsilon_0}\vec{a}}$





29-46

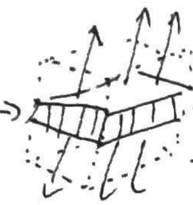




Translation + Reflection symmetry $\Rightarrow \vec{E} = E(z)\,\hat{z}$

Take a gaussian pillbox/prism w/ square base A:

(cross-section of large slab)



Centered at midplane, w/ top z above the midplane of the slab.

$\boxed{z > d/2}$

$$Q_{enc} = d \cdot \rho \cdot A$$

~~$\vec{E}\cdot\hat{n} = E(z)$; $\Phi = \Phi_{top} + \Phi_{bot} = 2A\,E(z)$; upper / lower ; $\Rightarrow |E(z)| = \frac{\Phi}{2A}$~~

$$\Phi_{top} = \int \vec{E}\cdot d\vec{A} = \int E(z)\,\hat{z}\cdot\hat{z} = E(z)\,A$$

$$\Phi_{bot} = \int \vec{E}\cdot d\vec{A} = \int E(-z)\,\hat{z}\cdot(-\hat{z}) = -E(-z)\,A$$

By symmetry (reflection), $E(-z) = -E(z)$

$$\Rightarrow \Phi = \Phi_{top} + \Phi_{bot} = 2A\,E(z)$$

$$E(z) = \Phi/2A \underset{G.L.}{=} \frac{Q_{enc}}{2\varepsilon_0 A} = \boxed{\frac{d\rho}{2\varepsilon_0}} \qquad z > d/2$$

$\boxed{z < d/2} \Rightarrow Q_{enc} = 2z\cdot\rho\cdot A$

$$\Rightarrow E(z) = \frac{2z\rho A}{2\varepsilon_0 A} = \boxed{\frac{z\rho}{\varepsilon_0}} \qquad z < d/2$$