2 Ideal Gas and Kinetic Theory

$k = 1.38 \times 10^{-23}$ J/K $\quad R = 8.314$ J/(mol K) $\quad 1$ atm $= 101325$ Pa

$pV = NkT \quad E_{int} = \frac{n_{dof}}{2} kT \quad p = \frac{1}{3}\rho v_{rms}^2$

$f(v) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} \quad v_{rms} = \sqrt{\frac{3kT}{m}} \quad v_{av} = \sqrt{\frac{8kT}{\pi m}} = 0.92\, v_{rms}$

2.a RHK Exercises

21.43 An air bubble of 20 cm^3 volume is at the bottom of a lake 40 m deep where temperature is 2 °C. The bubble rises to the surface, which is at a temperature of 27 °C. *(a)* What is the pressure as a function of depth below the surface of the lake? *(b)* Take the temperature of the bubble to be the same as that of the surrounding water and find its volume just before it reaches the surface. *(Round $\rho_{water} \approx 1$ g/cm^3, $g \approx 10$ m/s^2, and 1 atm $\approx 10^5$ Pa.)*

22.15 *(a)* Ten particles are moving with the following speeds: four at 200 m/s, two at 500 m/s, and four at 600 m/s. Calculate the average and root-mean-square speeds. Is $v_{rms} > v_{av}$? *(b)* Make up your own speed distribution for the ten particles and show that $v_{rms} \geq v_{av}$ for your distribution. *(c)* Under what condition (if any) does $v_{rms} = v_{av}$?

22.23 *(a)* Compute the temperatures at which the rms speed is equal to the speed of escape from the surface of the Earth (≈ 11 km/s) for molecular hydrogen (2 g/mol) and for molecular oxygen (16 g/mol). *(b)* Do the same for the Moon, $v_{moon} = 2.4$ km/s. *(c)* The temperature high in the Earth's upper atmosphere is about 1000 K. Would you expect to find much hydrogen there? Much oxygen?

22.18+26 *(a)* Determine the most probable speed in an ideal gas, by setting $\mathrm{d}f(v)/\mathrm{d}v = 0$. *(b)* It is found that the most probable speed of molecules in a gas at temperature T_2 is the same as the rms speed of the molecules in this gas when its temperature is T_1. Calculate $T_2\,/T_1$.

2.b RHK Problems

21.17 Two vessels of volumes 1 L and 4 L contain monatomic Krypton gas and are connected by a thin tube. Initially, the vessels are at the same temperature, 27 °C, and the same pressure, 1.6 atm. The larger vessel is then heated to 127 °C while the smaller one remains at 27 °*(a)* Calculate the final pressure. *(b)* Calculate the change in the internal energy of the gas.

22.2 Dalton's law states that when mixtures of gases having no chemical interaction are present together in a vessel, *(a)* the pressure exerted by each constituent at a given temperature is the same as it would exert if it alone filled the whole vessel, and *(b)* that the total pressure is equal to the sum of the partial pressures of each gas. Derive the two parts of this law from kinetic theory. *(Hint: $p_i = \frac{1}{3}\rho v_{rms,i}^2$)*

22.7 Two containers are at the same temperature. The first contains gas at pressure p_1 whose molecules have mass m_1 with a root-mean-square speed $v_{rms,1}$. The second contains molecules of mass m_2 at pressure $2p_1$ that have an average speed $v_{av,2} = 2v_{rms,1}$. Find the ratio m_1/m_2.

22.8 A gas, not necessarily in thermal equilibrium, consists of N particles. The speed distribution $f(v)$ is not necessarily Maxwellian. *(a)* Show that $v_{rms} \geq v_{av}$ regardless of the distribution of speeds. *Hint: Consider* $\left\langle (v - v_{av})^2 \right\rangle$ *(b)* When would the equality hold?

22.9 *(Modified)* In this problem we consider the "fake Maxwell-Boltzmann" speed distribution

$$g(v) = \begin{cases} Av^2 & v \leq v_0 \\ 0 & v > v_0 \end{cases} \tag{2.1}$$

where A and v_0 are constants that may depend on temperature, mass, etc.

(a) Make a plot of $g(v)$.

(b) Determine the appropriate A such that $g(v)$ agrees with the true Maxwell-Boltzmann distribution $f(v)$ in the limit $v \to 0$.

(c) Given the A you just calculated, find the v_0 that fixes the normalization $\int_0^\infty g(v)\,\mathrm{d}v = N$.

(d) Calculate v_{av} and v_{rms}. Which one is greater?

(e) Calculate the standard deviation of speeds, $\sigma = \sqrt{\langle v^2 \rangle - \langle v \rangle^2}$. *(For d+e, you can leave things in terms of A and v_0)*

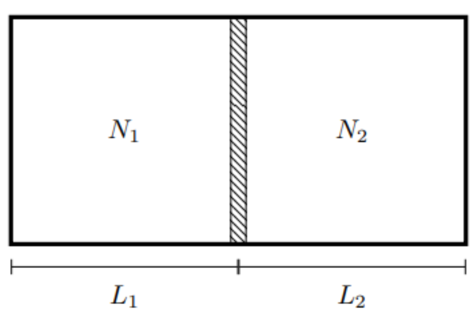
22.13 Consider a gas at temperature T occupying a volume V to consist of a mixture of atoms - namely, N_A atoms of mass m_A each having an rms speed v_A and N_B atoms of mass m_B each having an rms speed v_B.

(a) Give an expression for the total pressure exerted by the gas.

(b) Suppose now that $N_A = N_B$ and that the different atoms combine at constant volume to form molecules of mass $m_A + m_B$. Once the temperature returns to its original value, what would be the ratio of the pressure after combination to the pressure before?

2.c Other

1. A box of length 1 m and cross-sectional area A has a movable partition inside it. There are $N_1 = 3 \times 10^{23}$ molecules on the left and $N_2 = 2 \times 10^{23}$ molecules on the right. The gas on both sides is in thermal equilibrium at the same temperature T. When the partition settles down to its final position, find the length L_1 and L_2 of the left and right sides of the box.



2.d Bonus

1. In this bonus problem we do part of the derivation for the Maxwell speed distribution, and review some concepts from multivariable calculus.

(a) Calculate $I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$ by considering

$$I^2 = \left(\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2}\right)\left(\int_{-\infty}^{\infty} \mathrm{d}y\, e^{-y^2}\right) \tag{2.2}$$

and going to 2d polar coordinates, $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\phi$. Also, show that $\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x = I/\sqrt{a}$.

According to statistical mechanics *(beyond the scope of this course!)*, the probability distribution on the *full space of coordinates* is simply proportional to the Boltzmann factor $e^{-E/kT}$. For the ideal gas, this means

$$f(v_x, v_y, v_z) = \frac{1}{Z}\exp\Big[-\frac{m}{2kT}(v_x^2 + v_y^2 + v_z^2)\Big] \tag{2.3}$$

where v_x, v_y, v_z are the components of a gas particles *velocity* (not speed!), and Z is a normalization factor, so that $\int \mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z\, f(v_x, v_y, v_z) = 1$.

(b) Show that:

$$\frac{1}{Z} = \left(\frac{m}{\sqrt{2\pi kT}}\right)^3 \tag{2.4}$$

Recall that in 3d, we can go from cartesian coordinates (x, y, z) to spherical coordinates (ρ, θ, ϕ), where $\rho = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin, and θ, ϕ are the polar and azimuthal angles. The differential volume element in these coordinates is

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\,\mathrm{d}\rho\,\mathrm{d}\Omega = \rho^2\,\mathrm{d}\rho\,(\sin\theta\;\mathrm{d}\theta\mathrm{d}\phi) \tag{2.5}$$

We can just as well go to spherical coordinates in velocity space,

$$\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z = v^2\,\mathrm{d}v\,(\sin\theta\;\mathrm{d}\theta\mathrm{d}\phi) \tag{2.6}$$

(c) Write $f(v_x, v_y, v_z)\,\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z$ in terms of the spherical velocity coordinates.

(d) Integrate over the angle coordinates to get the probability distribution $\tilde{f}(v)$ for the speed v. I.e.,

$$\int_{v_0}^{v_0+\delta v} \tilde{f}(v)\,\mathrm{d}v \tag{2.7}$$

should give the probability that the molecule has speed between v_0 and $v_0 + \delta v$. *(Think "spherical shell")*