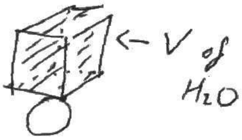
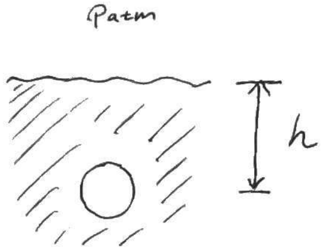
2. Kinetic Theory Solutions



21.43



(a) $p(h) = p_{atm} + p_{H_2O}$

$p_{H_2O} = F/A$ $\qquad F = Mg = (\rho V g) = \rho h A g$

$= \rho g h$ $\qquad \rho g = 1\ g/cm^3 \cdot 10\ Kg/ms^2$

$= 1000 \cdot 10\ Pa/m$

$= 10^4\ Pa/m$

$p(h) = p_{atm} + \rho g h$

$= 10^5 Pa + (10^4\ Pa/m)\, h \qquad p_{atm} =$

(b) $T_1 = 2°C = 275\,K \qquad p_1 = 10^5 + 40 \cdot 10^4 = 5 \cdot 10^5\ Pa$

$T_2 = 27°C = 300\,K \qquad p_2 = 1 \times 10^5\ Pa$

$pV = \underbrace{N k}_{const.} T \quad \rightarrow \quad V_2 = V_1 \cdot \frac{T_2}{T_1} \cdot \frac{p_1}{p_2}$

$= 20 \cdot \frac{300}{275} \cdot \frac{5}{1} = \boxed{110\ cm^3}$

22.15

(a)

v	200	500	600
$N(v)$	4	2	4

$\Rightarrow N_{tot} = 4+2+4 = 10$

$v_{av} = \frac{1}{N_{tot}} \sum v \cdot N(v) = \frac{1}{10}(4 \cdot 200 + 2 \cdot 500 + 4 \cdot 600)$ $\qquad 800 + 1000 + 2400 = 4200$

$= \cancel{320}\ 420\ m/s$

$v_{rms}^2 = \frac{1}{N_{tot}} \sum v^2 N(v) = \frac{1}{10}(4 \cdot 40000 + 2 \cdot 250000 + 4 \cdot 360000)$

$= 21 \cdot 10^4\ m^2/s^2$

$\Rightarrow v_{rms} = \sqrt{21} \cdot 10^2\ m/s \sim 460\ m/s$

$v_{rms} > v_{av}$

(b)

v	100	1000
$N(v)$	9	1

$\Rightarrow v_{av} = \frac{1}{10}(900 + 1000) = 190\ m/s$

$v_{rms}^2 = \frac{1}{10}(9 \cdot 10^4 + 10^6) = 9000 + 100\,000$

$v_{rms} > v_{av}$ $\qquad = 109\,000\ m^2/s^2 = 10.9 \times 10^4\ m^2/s^2$

$v_{rms} = \sqrt{v_{rms}^2} \simeq \sqrt{10.9} \times 100\ m/s \sim 330\ m/s$

(c) $v_{rms} = v_{av}$ if every molecule has the <u>same</u> speed v_0

$\Rightarrow v_{av} = \frac{1}{N} \sum v_0 = v_0$

$v_{rms}^2 = \frac{1}{N} \sum v_0^2 = \frac{1}{N}(N v_0^2) = v_0^2 \Rightarrow v_{rms} = v_0$

22.23

(a) $v_{rms} = \sqrt{\frac{3kT}{m}}$ $\qquad R = N_A k, \quad \underset{\text{(molar mass)}}{M} = N_a m \Rightarrow \frac{R}{M} = \frac{k}{m}$

$= \sqrt{\frac{3RT}{M}}$ $\qquad R \simeq 8 \ ^{J}/_{K\cdot mol}$

$J = kg \ ^{m^2}/_{s^2}$

$v_{esc} = v_{rms} \Rightarrow \frac{3RT}{M} = v_{esc}^2, \quad T = \frac{M v_{esc}^2}{3R}$

$M/R|_{H_2} = \cancel{2/8 \ 3/4} \ \frac{2 \ g/mol}{8 \ J/K\cdot mol} = \frac{1}{4} \ \frac{g \ K}{J} = \frac{1}{4000} K \cdot \frac{s^2}{m^2}$

$M/R|_{O_2} = \frac{16}{8} \frac{g/mol}{J/K\cdot mol} = \frac{2}{1000} K \cdot s^2/m^2 = 8 \cdot (M/R|_{H_2})$

$v_{esc}^2 = (11 \ km/s)^2 = 121 \cdot 10^6 \ m^2/s^2$

$\Rightarrow T_{H_2} = \frac{1}{3\cdot 4000} \cdot 121 \cdot 10^6 \ K = \frac{121}{12} \cdot 10^3 K \sim 10^4 K$

$T_{O_2} = (16/2) T_{H_2} \cong 8 \cdot 10^4 K$

(b) $v_{esc, moon}^2 = (2.4 \ km/s)^2 = 5.8 \cdot 10^6 \ m^2/s^2$, $\quad 5.8/121 \cong \frac{6}{120} = \frac{1}{20} = 0.05 = 5/100$

$\Rightarrow T_{H_2, moon} = 0.05 \times T_{H_2, Earth} = 5 \cdot 10^2 K = 500 K$

$T_{O_2, moon} = \frac{5.8}{100} \cdot 10^4 K = 4000 K$

(c) See the "Atmosphere of Earth" wikipedia article, esp. "Stratification" section.

22.18 $f(v) = N \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 4\pi v^2 e^{\frac{-mv^2}{2kT}}$

(a) $0 = \frac{d}{dv} f(v) \Leftrightarrow 0 = \frac{d}{dv}\left[v^2 e^{-mv^2/2kT}\right]$

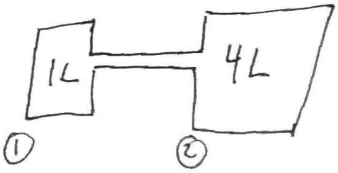
$= 2v e^{-mv^2/2kT} + v^2 \left[e^{-mv^2/2kT} \cdot \frac{-2mv}{2kT}\right]$

$= \underbrace{e^{-mv^2/2kT}}_{>0} \underbrace{\left[2v - \frac{mv^3}{kT}\right]}_{=0 \text{ if } 2 - \frac{mv^2}{kT} = 0} \Rightarrow \boxed{v = \sqrt{\frac{2kT}{m}}}$

(b) $v_{rms} = \sqrt{\frac{3kT}{m}}$

$\Rightarrow \sqrt{\frac{2kT_1}{m}} = \sqrt{\frac{3kT_2}{m}} \Rightarrow 2T_1 = 3T_2 \Rightarrow \boxed{T_2/T_1 = 3/2}$

21.17 (a)



$T_0 = 300K,\ p_0 = 1.6\ atm$

$T_1' = 300K,\ T_2' = 400K$

(a) The total # of atoms is const.

$$N_{tot} = N_1 + N_2 = \frac{V_1 p_0}{kT_0} + \frac{V_2 p_0}{kT_0} = (V_1+V_2)\frac{p_0}{kT_0}$$

After heating, equalize to pressure p'

$$N_{tot}' = N_1' + N_2' = \frac{V_1 p'}{kT_0} + \frac{V_2 p'}{kT_2'}$$

$$= \left(\frac{V_1}{kT_0} + \frac{V_2}{kT_2'}\right)p'$$

$$N_{tot} = N_{tot}' \Rightarrow (V_1+V_2)\frac{p_0}{kT_0} = \left(\frac{V_1}{kT_0} + \frac{V_2}{kT_2}\right)p'$$

$$\left(\frac{V_1}{T_0} + \frac{V_2}{T_0}\right)p_0 = \left(\frac{V_1}{T_0} + \frac{V_2}{T_2}\right)p'$$

$$(V_1 + V_2)p_0 = (V_1 + V_2(T_0/T_2))\,p'$$

$$\Rightarrow \boxed{p' = p_0 \cdot \frac{V_1 + V_2}{V_1 + (T_0/T_2)V_2}}$$

$T_0/T_2 = \frac{300}{400} = 3/4$

$$\Rightarrow p' = 1.6\ atm \cdot \frac{1+4}{1+(3/4)4} = 1.6 \cdot \frac{5}{4} = \boxed{2.0\ atm}$$

(b)

$d = 3$ (monatomic) $\rightarrow E_{int} = \frac{3}{2}NkT = \frac{3}{2}pV$

$$\Rightarrow E^{(0)}_{int(1+2)} = E_{int(1)} + E_{int(2)} = \frac{3}{2}p_0V_1 + \frac{3}{2}p_0V_2 = \frac{3}{2}p_0(V_1+V_2)$$

$$E'_{int(1+2)} = \frac{3}{2}p'V_1 + \frac{3}{2}p'V_2 = \frac{3}{2}p'(V_1+V_2)$$

$$\Rightarrow \Delta E = E' - E^{(0)} = \boxed{\frac{3}{2}(V_1+V_2)(p'-p_0)}$$

$$= \frac{3}{2}(5\ L)(0.4\ atm)\frac{10^5\ Pa}{1\ atm}\cdot\frac{1\ m^3}{1000\ L}$$

$$= 3/1000 \cdot 10^5 = \boxed{300\ J}$$

22.7

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

$$v_{av} = \sqrt{\frac{8kT}{\pi m}}$$

$$\frac{v_{av,2}^2}{v_{rms,1}^2} = \left(\frac{8kT}{\pi m_2}\right)\cdot\left(\frac{3kT}{m_1}\right)^{-1}$$

$$= \frac{8}{3\pi}\ m_1/m_2$$

$$\Rightarrow \frac{m_1}{m_2} = \frac{3\pi}{8}\left(\frac{v_{av,2}^2}{v_{rms,1}^2}\right) = \frac{3\pi}{8}\left(\frac{v_{av,2}}{v_{rms,1}}\right)^2 = \frac{3\pi}{8}\cdot(2)^2 = \frac{3\pi}{2} \approx \boxed{4.7}$$

(Note: Although problem gave pressures, solution doesn't actually need them)

22.8 $f(v)$ generic. Denote $\langle g(v) \rangle = \frac{1}{N}\int_0^\infty dv\ f(v)\,g(v)$

(a) $v_{av} = \langle v \rangle$, $v_{rms}^2 = \langle v^2 \rangle$

↳ (Note this just a #, not a function of v,
so $\langle v_{av} \rangle = \frac{1}{N}\int_0^\infty dv\ v_{av}\, f(v) = \frac{v_{av}}{N}\int_0^\infty dv\ f(v)$
$= v_{av}$)

Consider $\langle (v - v_{av})^2 \rangle = \langle v^2 - 2 v_{av} v + v_{av}^2 \rangle$

$= \langle v^2 \rangle - \langle 2 v_{av} v \rangle + \langle v_{av}^2 \rangle$

$= \langle v^2 \rangle - 2 v_{av} \langle v \rangle + v_{av}^2$

$= \langle v^2 \rangle - 2 v_{av} \cdot v_{av} + v_{av}^2$

$= \langle v^2 \rangle - v_{av}^2 = v_{rms}^2 - v_{av}^2$

But $(v - v_{av})^2 \geq 0$ (since it's squared)

so $\langle (v - v_{av})^2 \rangle = \frac{1}{N}\int_0^\infty dv\ \underbrace{f(v)}_{\geq 0}\ \underbrace{(v - v_{av})^2}_{\geq 0} \geq 0$

$\Rightarrow\ 0 \leq \langle (v - v_{av})^2 \rangle = v_{rms}^2 - v_{av}^2 \Rightarrow v_{av}^2 \leq v_{rms}^2$

$\boxed{v_{av} \leq v_{rms}}$

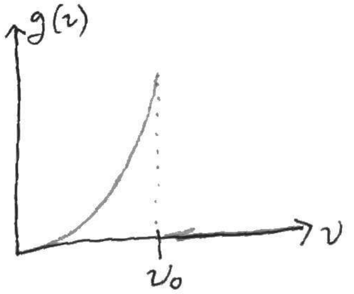
(b) The equality holds if $0 = \langle (v - v_{av})^2 \rangle$

$\Leftrightarrow\ 0 = (v - v_{av})^2$ for all v

that is, $v = v_{av}$ – every molecule is at the <u>same</u> speed.

22.49 $\quad g(v) = \begin{cases} A v^2 & v \leq v_0 \\ 0 & v > v_0 \end{cases}$, $\quad f(v) = N\left(\sqrt{\frac{m}{2\pi kT}}\right)^3 4\pi v^2 e^{-\frac{mv^2}{2kT}}$

(a)



(b) We want

$$f(v) \approx g(v) \quad \text{as} \quad v \to 0$$

(So we only need to consider $v \leq v_0$ part)

In other words:

$$\boxed{\lim_{v\to 0} \frac{f(v)}{g(v)} = 1}$$ ✱ Key step

$$\frac{f(v)}{g(v)} = \frac{N\left(\sqrt{m/2\pi kT}\right)^3 4\pi v^2 e^{-mv^2/2kT}}{A v^2}$$

$$= \underbrace{\frac{1}{A}\left(N\left(\sqrt{m/2\pi kT}\right)^3 4\pi\right)}_{\#,\ \text{independent of } v} e^{-mv^2/2kT}$$

$$\Rightarrow \lim_{v\to 0} \frac{f(v)}{g(v)} = \frac{1}{A}\left(N\left(\sqrt{m/2\pi kT}\right)^3 4\pi\right) \underbrace{\lim_{v\to 0} e^{-mv^2/2kT}}_{1} \qquad (e^0 = 1)$$

$$= \frac{1}{A}(\cdots) = 1$$

$$\Rightarrow \boxed{A = N\left(\sqrt{m/2\pi kT}\right)^3 4\pi}$$

(c)

$$N = \int_0^\infty g(v)\,dv = \int_0^{v_0} g(v)\,dv + \int_{v_0}^\infty g(v)\,dv$$

$$= \int_0^{v_0} Av^2\,dv + \int_{v_0}^\infty (0)\,dv$$

$$= A\int_0^{v_0} v^2\,dv + 0$$

$$N = A\, v_0^3/3$$

$$\Rightarrow v_0 = \left(\frac{3N}{A}\right)^{1/3} = \left(\frac{3N}{N(\sqrt{\cdots})^3 4\pi}\right)^{1/3} = \left(\frac{3}{4\pi}\left[\sqrt{\frac{2\pi kT}{m}}\right]^3\right)^{1/3}$$

$$\boxed{v_0 = (3/4\pi)^{1/3}\sqrt{\frac{2\pi kT}{m}}}$$

$\left(\approx \sqrt{\frac{2.4\,kT}{m}}\right.$, so close to v_{rms} of true maxwell dist.$\left.\right)$

22.9

(d)

$$v_{av} = \frac{1}{N}\int_0^\infty g(v)\cdot v\, dv = \frac{1}{N}\int_0^{v_0} v\, g(v)\, dv + \underbrace{\frac{1}{N}\int_{v_0}^\infty v\, g(v)\, dv}_{0}$$

$$= \frac{1}{N}\int_0^{v_0} v\,(Av^2)\, dv$$

$$= \frac{A}{N}\int_0^{v_0} v^3\, dv = \frac{A}{4N} v_0^4 \qquad = \frac{v_0}{4}\frac{Av_0^3}{N} = \frac{3v_0}{4}$$

$$v_{rms}^2 = \frac{1}{N}\int_0^\infty g(v)\, v^2\, dv = \frac{1}{N}\int_0^{v_0} v^2 (Av^2)\, dv$$

$$= \frac{A}{N}\int_0^{v_0} v^4\, dv = \frac{A}{5N} v_0^5 \quad = \frac{3v_0^2}{5}$$

If we used earlier:

$$\Rightarrow v_{rms} = \sqrt{A/5N\; v_0^5}$$

To compare v_{rms} to v_{av},

look at the ratio $v_{rms}^2/v_{av}^2 = \frac{A}{5N} v_0^5 \Big/ \left(\frac{A^2}{4^2N^2} v_0^8\right)$

$$= \frac{16}{5}\cdot\frac{A}{N}\cdot\frac{N^2}{A^2}\cdot v_0^{-3} = \frac{16}{5}\;\frac{N}{Av_0^3}$$

In part (c) showed that $N = \frac{A v_0^3}{3} \Rightarrow \frac{N}{Av_0^3} = \frac{1}{3}$

$$\Rightarrow \frac{v_{rms}^2}{v_{av}^2} = \frac{16}{5}\cdot\frac{1}{3} = \frac{16}{15} > 1$$

$v_{rms}^2/v_{av}^2 > 1$, so $v_{rms}/v_{av} > 1 \Rightarrow \boxed{v_{rms} > v_{av}}$

(e)

$$\sigma^2 = \langle v^2\rangle - \langle v\rangle^2 = v_{rms}^2 - v_{av}^2$$

$$= \frac{A}{5N} v_0^5 - \frac{A^2}{16N^2} v_0^8 = \frac{Av_0^5}{N}\left(\frac{1}{5} - \frac{1}{16}\overbrace{v_0^3\frac{A}{N}}^{3}\right)$$

$$= \frac{Av_0^5}{N}\left(\frac{1}{5} - \frac{3}{16}\right) = \frac{1}{80}\frac{A}{N} v_0^5 = \frac{1}{80}\left(\frac{Av_0^3}{N}\right) v_0^2$$

$$= \boxed{\frac{3}{80} v_0^2}$$

22.13 $\quad T, V \quad \begin{matrix} N_A, m_A, v_A \\ N_B, m_B, v_B \end{matrix}$

(a) $$P_{tot} = P_A + P_B = \frac{N_A kT}{V} + \frac{N_B kT}{V} = (N_A + N_B)\frac{kT}{V}$$

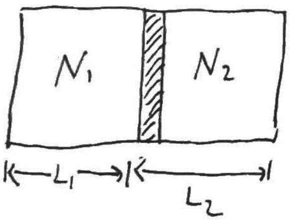
(b) $$P_{tot}' = \frac{\overset{= N_A = N_B}{N'}\, k \overset{\text{same}}{T'}}{\underset{\text{same}}{V'}} = \frac{N_A k T'}{V} = \frac{N_A kT}{V}$$

$$\Rightarrow P_{tot}'/P_{tot} = \frac{N_A kT'}{V} \Big/ \left((N_A+N_B)\frac{kT}{V}\right) = \underbrace{\frac{N_A}{N_A+N_B}}_{\frac{1}{2}} \underbrace{\frac{T'}{T}}_{1} = \boxed{\frac{1}{2}}$$

(Because we've returned to the same temperature $T' = T$, and the volume is the same $V' = V$, all that can change the pressure is the # of molecules changing)

So m, v actually don't matter.

Other 1



$$pV = NkT$$

In equilibrium, $T_1 = T_2 = T$

and $\quad p_1 = p_2 = p \quad$ (else there would be a net force on the partition)

$$\Rightarrow \frac{V_i}{N_i} = \frac{kT}{p} \quad \text{same for } i = 1, 2$$

$$\Rightarrow \frac{V_1}{N_1} = \frac{V_2}{N_2} \Rightarrow \frac{V_1}{V_2} = \frac{N_1}{N_2} = \frac{3}{2}$$

$V_i = L_i \cdot A \quad$ (same cross-section area for both)

$$\Rightarrow L_1/L_2 = V_1/V_2 = 3/2 \quad \Rightarrow \frac{L_1}{L_1+L_2} = \frac{3}{5}, \quad \frac{L_2}{L_1+L_2} = \frac{2}{5}$$

$$\rightarrow \boxed{L_1 = 0.6\ \text{m}, \quad L_2 = 0.4\ \text{m}}$$

Bonus

(a) $$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

← Just a relabelling, $x \to \quad \to y$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$$

$$= \underbrace{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy}_{\text{Integral over whole xy-plane}} \; e^{-x^2} e^{-y^2} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-(x^2+y^2)}$$

Integral over whole xy-plane $= \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta$ in polar coordinates

$$\Rightarrow I^2 = \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta \left[e^{-r^2}\right] \qquad x^2 + y^2 = r^2$$

$$= \int_0^{\infty} dr \int_0^{2\pi} d\theta \; r e^{-r^2}$$

$$= 2\pi \int_0^{\infty} dr \; r e^{-r^2} \qquad \leftarrow \text{(No } \theta\text{-dependence)}$$

$$= 2\pi \left[-\frac{1}{2} e^{-r^2}\right]_0^{\infty} \qquad \leftarrow \left(\frac{d}{dr} e^{-r^2} = e^{-r^2}\cdot(-2r) = -2re^{-r^2} \checkmark\right)$$

$$= 2\pi\left[0 + \tfrac{1}{2}\right] = \pi$$

$$\Rightarrow \boxed{I = \sqrt{\pi}}$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \int_{-\infty}^{\infty} \left(\frac{d(\sqrt{a}\,x)}{\sqrt{a}}\right) \exp\left(-(\sqrt{a}\,x)^2\right)$$

$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} du\; e^{-u^2} = I/\sqrt{a} = \sqrt{\pi/a}$$

← $u = x\sqrt{a}$ substitution.
$du = \sqrt{a}\,dx$
$\Rightarrow dx = du/\sqrt{a}$, $(ax^2 = u^2)$

(b) $$f(v_x, v_y, v_z) = \frac{1}{Z} \exp\left(-\frac{m}{2kT}\left(v_x^2 + v_y^2 + v_z^2\right)\right)$$

$$= \frac{1}{Z} e^{\frac{-m v_x^2}{2kT}} e^{\frac{-m v_y^2}{2kT}} e^{\frac{-m v_z^2}{2kT}}$$

$$\int dv_x dv_y dv_z\, f(v_x, v_y, v_z) = \frac{1}{Z} \int dv_x dv_y dv_z \; e^{-\frac{m v_x^2}{2kT}} e^{-\frac{m v_y^2}{2kT}} e^{-\frac{m v_z^2}{2kT}}$$

$$= \frac{1}{Z}\left(\int dv_x\, e^{-m v_x^2/2kT}\right)\left(\int dv_y\, e^{-m v_y^2/2kT}\right)\left(\int dv_z\, e^{-m v_z^2/2kT}\right)$$

Use $a = +m/2kT \Rightarrow \; = \frac{1}{Z}\left(\sqrt{\frac{2\pi kT}{m}}\right)\left(\sqrt{\frac{\pi kT}{m}}\right)\left(\sqrt{\frac{\pi kT}{m}}\right)$

$$\Rightarrow \boxed{\frac{1}{Z} = \left(\sqrt{\frac{m}{2\pi kT}}\right)^3}$$

(In order for $\frac{1}{Z}\int d f(v_x, v_y, v_z) = 1$)

(c) $$f(v_x, v_y, v_z)\, dv_x\, dv_y\, dv_z = \frac{1}{Z} \exp\left[-\frac{mv^2}{2KT}\right] \cdot v^2\, dv\, (\sin\theta\, d\theta\, d\phi)$$

(d) $$\Rightarrow \int f(v_x, v_y, v_z)\, dv_x\, dv_y\, dv_z = \frac{1}{Z} \int \exp\left[\frac{-mv^2}{2KT}\right] v^2\, dv\, (\sin\theta\, d\theta\, d\phi)$$

$$= \frac{1}{Z} \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\left(\int_0^{\infty} dv \cdot v^2 \exp\left[\frac{-mv^2}{2KT}\right]\right)}_{\text{Independent of } \theta, \phi}$$

$$\cancel{= \frac{1}{Z}\int_0^{\pi}} = \int_0^{\infty} dv\, \frac{4\pi}{Z}\, v^2 \exp\left[\frac{-mv^2}{2KT}\right]$$

$$= \int_0^{\infty} dv\, \tilde{f}(v)$$

$$\Rightarrow \boxed{\hat{f}(v) = \frac{1}{Z}(4\pi v^2) \exp\left[\frac{-mv^2}{2KT}\right]}$$