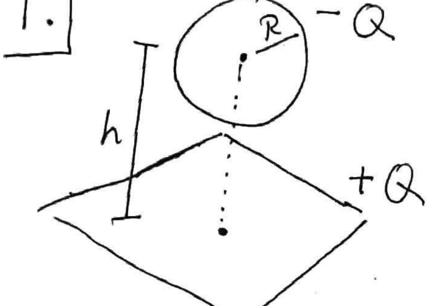
1.

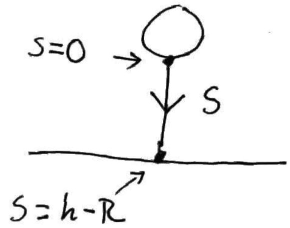


(a) Place charge $+Q$ on sheet, $-Q$ on sphere.

Superposition $\Rightarrow \vec{E} = \vec{E}_{sphere} + \vec{E}_{plane}$

$$\vec{E}_{sphere} = \frac{1}{4\pi\varepsilon_0}\frac{(-Q)}{r^2}\hat{r}$$

$$\vec{E}_{sheet} = \pm\frac{\sigma}{2\varepsilon_0}\hat{z}, \quad \sigma = Q/A$$



Focus on vertical line segment $s \in [0, h-R]$ connecting surface of sphere and sheet.

(Always want to integrate from − to + plates)

$$\Rightarrow \vec{E}_{sphere} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}(-\hat{z}) = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{(R+s)^2}\hat{s}$$

$$\vec{E}_{sheet} = +\frac{\sigma}{2\varepsilon_0}\hat{z} = -\frac{\sigma}{2\varepsilon_0}\hat{s}$$

$$\Rightarrow \vec{E} = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{(R+s)^2} + \frac{Q}{2A\varepsilon_0}\right)(-\hat{s})$$

$$\hookrightarrow \Delta V = -\int_-^+ \vec{E}\cdot d\vec{s} = +\int_0^{h-R}\left[\frac{1}{4\pi\varepsilon_0}\frac{Q}{(R+s)^2} + \frac{Q}{2A\varepsilon_0}\right]ds$$

$$= \frac{Q}{2\varepsilon_0}\int_0^{h-R}\left[\frac{1}{2\pi}\frac{1}{(R+s)^2} + \frac{1}{A}\right]ds$$

$$= \frac{Q}{2\varepsilon_0}\left[\frac{1}{2\pi}\left(\frac{1}{R} - \frac{1}{h}\right) + \frac{h-R}{A}\right]$$

$$= \frac{Q(h-R)}{2\varepsilon_0}\left[\frac{1}{2\pi}\frac{1}{Rh} + \frac{1}{A}\right] \geq 0 \qquad (h-R \geq 0)$$

$$\Rightarrow \boxed{C = \frac{Q}{\Delta V} = \frac{\varepsilon_0}{2(h-R)}\left[\frac{1}{2\pi Rh} + \frac{1}{A}\right]^{-1}}$$

(b)
$$\lim_{A\to\infty} C = \frac{\varepsilon_0}{2(h-R)}\left[\frac{1}{2\pi Rh} + 0\right]^{-1} = \varepsilon_0 \cdot \frac{2\pi Rh}{2(h-R)} = \varepsilon_0 \cdot \frac{\pi Rh}{(h-R)}$$

$\hookrightarrow$ $h = R + d$, d small $\Rightarrow \boxed{C \approx \varepsilon_0 \cdot \frac{\pi R^2}{d}}$

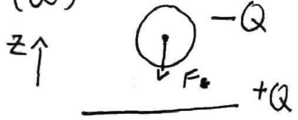
Looks like a <u>circular</u> parallel plate capacitor.

(c) R, Q fixed, $A \to \infty \Rightarrow C = \varepsilon_0 \dfrac{\pi R h}{(h-R)}$

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2\pi\varepsilon_0}\frac{Q^2}{R}\frac{h-R}{h} = \boxed{\frac{1}{2\pi\varepsilon_0}\frac{Q^2}{R}\left(1-\frac{R}{h}\right)}$$

$$\frac{dU}{dh} = \frac{1}{2\pi\varepsilon_0}\frac{Q^2}{R}\frac{d}{dh}\left[1-R/h\right] = \frac{1}{2\pi\varepsilon_0}\frac{Q^2}{R}\left(0+\frac{R}{h^2}\right) = \boxed{\frac{1}{2\pi\varepsilon_0}\frac{Q^2}{h^2}}$$

(d)



Intuitively, there is an electrostatic <u>attraction</u> between the two (opposite charges).

Indeed: $F_z = -\dfrac{dU}{dh} = -\dfrac{1}{2\pi\varepsilon_0}\dfrac{Q^2}{h^2}$

To counteract this, need $\boxed{\vec{F}_{ext} = -F_z\,\hat{z} = +\dfrac{1}{2\pi\varepsilon_0}\dfrac{Q^2}{h^2}\,\hat{z}}$

2.) (a)

Gauss law $\Rightarrow \vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{r}$ $\quad a<r<b$

$$\Delta V = V_a - V_b = +\int_a^b \vec{E}\cdot d\vec{r}$$

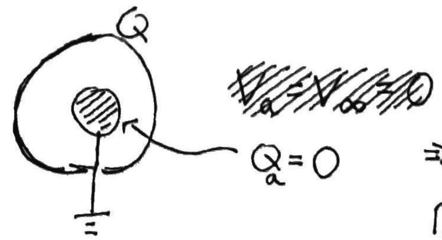
$$= \frac{1}{4\pi\varepsilon_0} Q \int_a^b \frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{b-a}{ab} > 0$$

$$\hookrightarrow \boxed{C = \frac{Q}{\Delta V} = \frac{4\pi ab}{b-a}\varepsilon_0}$$

(b)



$Q_a = 0$ $\Rightarrow E=0$ b/w the spheres.

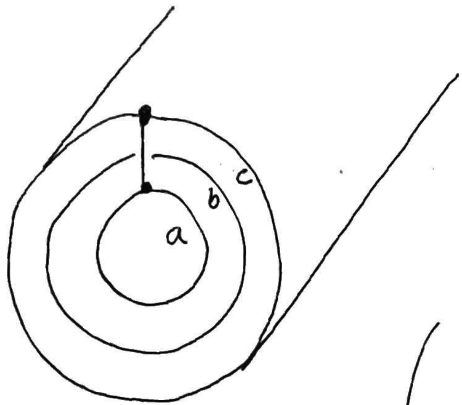
But $\Delta V \equiv V_b - V_\infty \neq 0$, (since $\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$ $\quad r>b$)

$$V_b - V_\infty = +\int_b^\infty \vec{E}\cdot d\vec{r} = \frac{1}{4\pi\varepsilon_0} Q\int_b^\infty \frac{1}{r^2}dr$$

$$= \frac{Q}{4\pi\varepsilon_0 b}$$

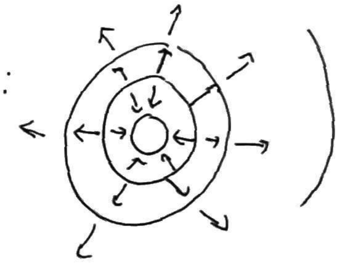
$$\Rightarrow \boxed{C = \frac{Q}{\Delta V} = 4\pi\varepsilon_0 b}$$

3



- Connecting wire $\Rightarrow V_a = V_c$

  $\lambda_b > 0$
- Charge conservation $\Rightarrow \lambda_a + \lambda_c = 0$

(Qualitatively expect field like: (so that $V_c - V_b = V_a - V_b$) )

- Cylindrical symmetry $\Rightarrow \vec{E} = E(r)\,\hat{r}$
- Gaussian cylinder radius r, length L

$$\Phi = \int_{S(r)} \vec{E}\cdot\hat{r}\,dA = E(r)\int_{S(r)} dA = (2\pi r L)\,E(r)$$

$$\Rightarrow E(r) = \frac{\Phi}{2\pi r L} = \frac{Q_{enc}}{2\pi\varepsilon_0 L}\,\frac{1}{r}$$

$$Q_{enc} = \begin{cases} 0 & r<a \\ \lambda_a L & a<r<b \\ (\lambda_a+\lambda_b)L & b<r<c \\ (\lambda_a+\lambda_b+\lambda_c)L = \lambda_b L & c<r \end{cases} \Rightarrow E(r) = \begin{cases} 0 \\ (\lambda_a/2\pi\varepsilon_0)\frac{1}{r} \\ (\lambda_a+\lambda_b)/(2\pi\varepsilon_0 r) \\ (\lambda_b/2\pi\varepsilon_0)\,1/r \end{cases}$$

- $V_c - V_a = 0$

$$V_c - V_a = -\int_a^c \vec{E}\cdot d\vec{r} = -\left(\int_a^b \vec{E}\cdot d\vec{r} + \int_b^c \vec{E}\cdot d\vec{r}\right) = (V_c - V_b) + (V_b - V_a)$$

$$= -\frac{\lambda_a}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} - \frac{\lambda_a+\lambda_b}{2\pi\varepsilon_0}\int_b^c \frac{dr}{r}$$

$$= -\frac{\lambda_a}{2\pi\varepsilon_0}\ln b/a - \frac{\lambda_a+\lambda_b}{2\pi\varepsilon_0}\ln c/b = 0$$

$$\Leftrightarrow \quad \lambda_a \ln b/a + (\lambda_a+\lambda_b)\ln c/b = 0$$

$$\lambda_a(\ln b/a + \ln c/b) + \lambda_b \ln c/b = 0$$

$$\lambda_a \ln c/a + \lambda_b \ln c/b = 0$$

$$\Rightarrow \boxed{\begin{aligned} \lambda_a &= -\lambda_b \frac{\ln c/b}{\ln c/a} \\ \lambda_c &= -\lambda_a = +\lambda_b \frac{\ln c/b}{\ln c/a} \end{aligned}}$$

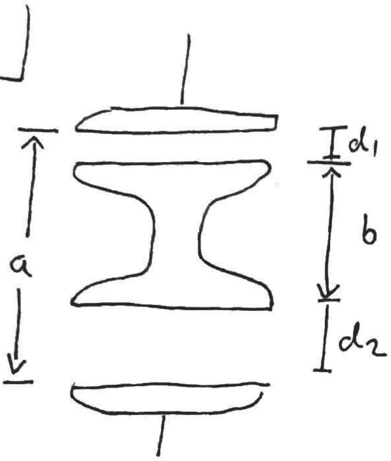
3) (Continued)

$$V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{r} = -\frac{\lambda_a}{2\pi\varepsilon_0} \ln {}^b\!/\!_a$$

Substitute $\lambda_a = -\lambda_b \dfrac{\ln {}^c\!/\!_b}{\ln {}^c\!/\!_a}$ $\Rightarrow$ $V_b - V_a = \dfrac{\lambda_b}{2\pi\varepsilon_0}\dfrac{(\ln {}^c\!/\!_b)(\ln {}^b\!/\!_a)}{\ln {}^c\!/\!_a} > 0$, since $c > b > a$

$$\Rightarrow \boxed{{}^C\!/\!_L = \frac{\lambda_b}{V_b - V_a} = 2\pi\varepsilon_0 \frac{\ln {}^c\!/\!_a}{(\ln {}^c\!/\!_b)(\ln {}^b\!/\!_a)}}$$

4)



This is two capacitors in series, ~~because the center~~

$$C^{-1} = C_1^{-1} + C_2^{-1}$$

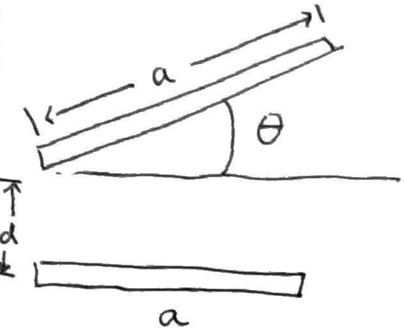
$$= \frac{d_1}{A\varepsilon_0} + \frac{d_2}{A\varepsilon_0} = \frac{1}{A\varepsilon_0}(d_1 + d_2)$$

From the diagram, see that $a = d_1 + b + d_2$

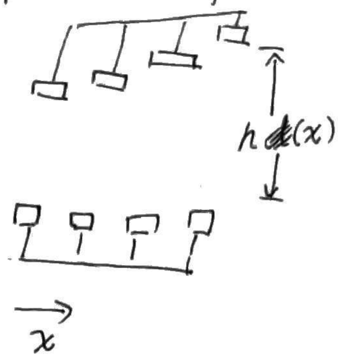
$$\Rightarrow d_1 + d_2 = a - b$$

$$\hookrightarrow C^{-1} = \frac{a-b}{A\varepsilon_0} \rightarrow \boxed{C = \frac{A\varepsilon_0}{a-b}}$$

5]



Split into infinitesimal capacitors in <u>parallel</u>:



Each infinitesimal one is parallel plate, with

area $a \cdot dx$

thickness $h(x) = d + x\tan\theta$

$\approx d + x\theta$

$$\Rightarrow dC = \frac{\varepsilon_0\, a\, dx}{d + x\theta}$$

$$\text{Parallel} \Rightarrow C = \int dC = \int_0^a \frac{\varepsilon_0\, a\, dx}{d + x\theta} = \frac{\varepsilon_0 a}{\theta}\left[\ln(d + x\theta)\right]_0^a$$

$$= \frac{\varepsilon_0 a}{\theta}\left(\ln(d + \theta a) - \ln(d)\right)$$

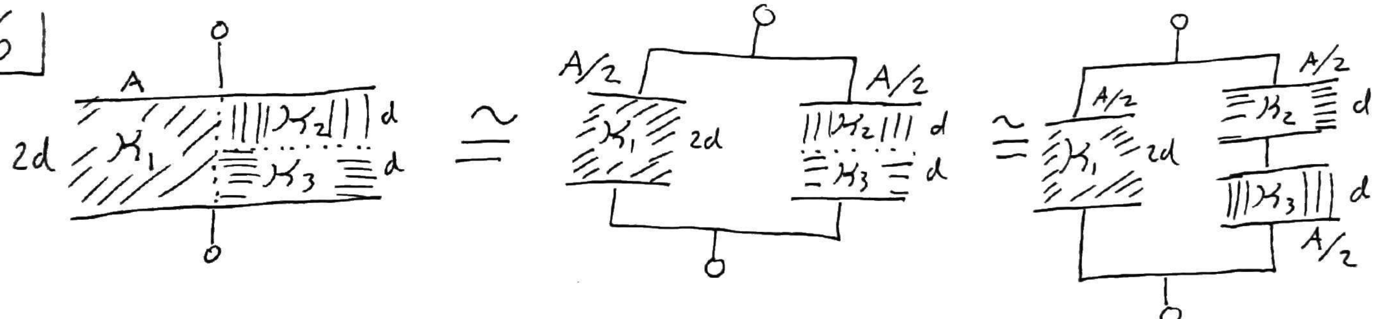
(These bounds are good as long as θ small, i.e. horizontal width $= a\cos\theta \approx a$)

$$= \frac{\varepsilon_0 a}{\theta}\ln\left(1 + \frac{\theta a}{d}\right) \qquad \ln(1+x) = x - \frac{x^2}{2} + \dots$$

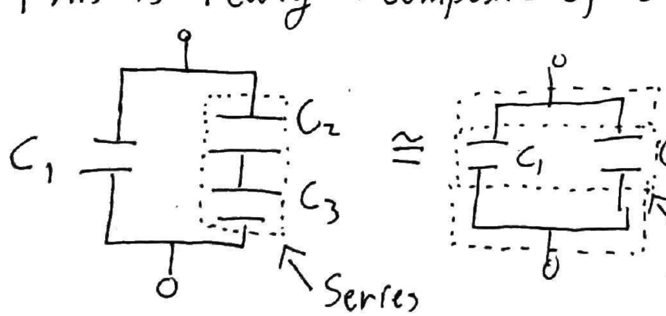
$$\simeq \frac{\varepsilon_0 a}{\theta}\left(\frac{\theta a}{d} - \frac{\theta^2 a^2}{2d^2}\right)$$

$$= \boxed{\frac{\varepsilon_0 a^2}{d}\left(1 - \frac{\theta a}{2d}\right)}$$

6]



This is really a composite of 3 capacitors:



$C_{23} = \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1}$

Parallel: $C = C_1 + C_{23}$

6| (Continued)

$$C = C_1 + C_{23}$$

$$C_1 = \frac{A(\kappa_1 \varepsilon_0)}{2 \cdot (2d)}, \quad C_2 = \frac{A \kappa_2 \varepsilon_0}{2d}, \quad C_3 = \frac{A \kappa_3 \varepsilon_0}{2d}$$

$$C_{23} = \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} = \frac{A \varepsilon_0}{2d}\left(\frac{1}{\kappa_2} + \frac{1}{\kappa_3}\right)^{-1}$$

$$= \frac{A \varepsilon_0}{2d}\left(\frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3}\right)$$

$$\boxed{C = C_1 + C_{23} = \frac{A \varepsilon_0}{2d}\left(\frac{\kappa_1}{2} + \frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3}\right)}$$