1 Temperature and Thermal Expansion

$$\Delta L = \alpha L_0\, \Delta T \qquad \Delta V = \beta V_0\, \Delta T \qquad \beta \approx 3\alpha$$
$$\alpha_{steel} = 11 \times 10^{-6}/K \qquad \alpha_{brass} = 19 \times 10^{-6}/K \qquad \alpha_{alum.} = 23 \times 10^{-6}/K$$

1.a RHK Questions

21. What are the dimensions of α, the coefficient of linear expansion? Does the value of α depend on the unit of length used? When Fahrenheit degrees are used instead of Celsius degrees as the unit of temperature change, does the numerical value of α change? If so, how? If not, prove it.

Answer: $[\alpha]$ = 1 / K. (Or 1 / °C.) The unit of length does not matter, but if using Fahrenheit then

$$\alpha = x\frac{1}{C} = x\frac{1}{C} \times \frac{5C}{9F} = \frac{5x}{9}\frac{1}{F} \tag{1.1}$$

Since a degree Fahrenheit is "smaller" than a degree Celsius, heating a material by 1 °F won't cause it to expand as much.

22. A metal ball can pass through a metal ring. When the ball is heated, however, it gets stuck in the ring. What would happen if the ring, rather than the ball, were heated?

Answer: When the ring is heated, the whole thing "scales up" in size due to thermal expansion, including the hole. So the ball will pass through easily.

27. Water expands when it freezes. Can we define a coefficient of volume expansion for the freezing process?

Answer: During the freezing process, the volume of water changes but the temperature stays constant. So if we tried the usual

$$\beta = \frac{\Delta V}{V_0}\frac{1}{\Delta T} \tag{1.2}$$

we would get $\beta = \infty$ (since $\Delta T = 0$).

29. Does the change in volume of an object when its temperature is raised depend on whether the object has cavities inside, other things being equal?

Answer: It doesn't matter. Divide up the object into a bunch of tiny cubes, labeled by i. Then

$$\Delta V = \sum_i \Delta V^{(i)} = \sum_i \left(\beta V_0^{(i)} \Delta T\right) = \beta \Delta T \sum_i V_0^{(i)} = \beta \Delta T \cdot V_0 \tag{1.3}$$

regardless of how these cubes are arranged, i.e. the shape of the object.

1.b RHK Exercises

15. Steel railroad tracks are laid when the temperature is -5.0 °C. A standard section of rail is then 12.0 m long. What gap should be left between rail sections so that there is no compression when the temperature gets as high as 42 °C?

Answer: Draw a picture. From it you should see that the gap size x needs to be

$$x \geq \frac{\Delta L}{2} + \frac{\Delta L}{2} = \Delta L \tag{1.4}$$

in order for the rail tracks not to run into each other. Now we just compute

$$\Delta L = \alpha L_0 \Delta T = (11 \times 10^{-6}/^\circ\mathrm{C})(12.0\ \mathrm{m})(47^\circ\mathrm{C}) = 6204 \times 10^{-6}\ \mathrm{m} = \boxed{6.2\ \mathrm{mm}} \tag{1.5}$$

20. Soon after the Earth formed, heat released by the decay of radioactive elements raised the average internal temperature from 300 to 3000 K, at about which value it remains today. Assuming an average coefficient of volume expansion of $3.2 \times 10^{-5}\ \mathrm{K}^{-1}$, by how much has the radius of the Earth increased since its formation? *(The current radius of the earth is 6400 km)*

Answer: Take the current volume of the earth as V_0, and consider the reverse cooling process,

$$V_{old} = (1 + \beta \Delta T) V_0 \tag{1.6}$$

where $\Delta T = -2700$ K. Then the old radius of the earth was (using $V = (4\pi/3) r^3$)

$$r_{old} = \left[\frac{3}{4\pi} V_{old}\right]^{1/3} = \left[\frac{3}{4\pi} V_0 (1 + \beta \Delta T)\right]^{1/3} = \left[\frac{3}{4\pi}\left(\frac{4\pi}{3} r_0^3\right)(1 + \beta \Delta T)\right]^{1/3} = [1 + \beta \Delta T]^{1/3} \cdot r_0 \tag{1.7}$$

in the last step, canceled the factors of $4\pi/3$ and pulled the r^3 out of the cube root. Now we just plug in,

$$r_{old} = \left[1 + (3.2 \times 10^{-5} \times 2700)\right]^{1/3} (6400\ \mathrm{km}) = (1.028) \times 6400\ \mathrm{km} = 6579\ \mathrm{km} \tag{1.8}$$

so $\Delta r = 179$ km. Alternatively, we could have considered linear expansion $\alpha \approx \beta/3$, which would give

$$\Delta r = (\beta/3) r_0 \Delta T = 3.2 \times 10^{-5} \times 2700 \times 6400\ \mathrm{km}/3 = 184\ \mathrm{km} \tag{1.9}$$

These are pretty close, and if we work to 2 significant figures (as we should since we only have 2 for β), then they are equivalent: $\boxed{180\ \mathrm{km}}$. Both approaches work because

$$[1 + \beta \Delta T]^{1/3} \cong 1 + \frac{\beta \Delta T}{3} - \frac{(\beta \Delta T)^2}{9} + \cdots \tag{1.10}$$

so $\alpha \approx \beta/3$ is a good approximation as long as $\beta \Delta T$ is small (it equals 0.086 in this case).

21. A rod is measured to be 20.05 cm long using a steel ruler at a room temperature of 20 °C. Both the rod and the ruler are placed in an oven at 270 °C, where the rod now measures 20.11 cm using the same rule. Calculate the coefficient of thermal expansion for the material of which the rod is made.

Answer: This is a tricky problem! We have to be careful because our measuring device is also expanding. The approach I advocated in class was to be careful about what "measuring on the ruler" really means. Let's use 1 for the ruler, and 2 for the rod, and imagine that our ruler is a meter stick i.e. it has 100 "ticks" on it that are supposed to measure 1 cm each. Then what we actually know from measuring the rod against the ruler is the ratios

$$\frac{L_2^{cold}}{L_1^{cold}} = \frac{20.05 \text{ ticks}}{100 \text{ ticks}}, \qquad \frac{L_2^{hot}}{L_1^{hot}} = \frac{20.11 \text{ ticks}}{100 \text{ ticks}} \tag{1.11}$$

Because the choice of 100 ticks was arbitrary, we would like to eliminate it. So we do:

$$\frac{L_2^{cold}}{L_1^{cold}} \times \frac{L_1^{hot}}{L_2^{hot}} = \frac{20.05 \text{ ticks}}{20.11 \text{ ticks}} = 0.997 = \frac{L_2^{cold}}{L_2^{hot}} \times \frac{L_1^{hot}}{L_1^{cold}} \tag{1.12}$$

Now we use $L_i^{hot}/L_i^{cold} = 1 + \alpha_i \Delta T$, to get

$$0.997 = \frac{1+\alpha_1 \Delta T}{1+\alpha_2 \Delta T} \tag{1.13}$$

and solve this for α_2:

$$1 + \alpha_2 \Delta T = \frac{20.11}{20.05}(1+\alpha_1 \Delta T) \implies \alpha_2 = \frac{1}{\Delta T}\left(-1 + \frac{20.11}{20.05}(1+\alpha_1 \Delta t)\right) \tag{1.14}$$

Rearranging slightly,

$$\alpha_2 = \frac{1}{\Delta T}\frac{20.11}{20.05}\left[(1 - \frac{20.05}{20.11}) + \alpha_1 \Delta T\right] \tag{1.15}$$

Plugging in the values for ΔT and α_1 gives

$$\alpha_2 = \frac{1}{250 \text{ K}}(1.0030)\left[0.0030 + 11 \times 10^{-6} \times 250\right] = \boxed{23 \times 10^{-6} / \text{ K}} \tag{1.16}$$

There is another way to solve this problem which is maybe a bit simpler. *(Thanks to Yun-Hsuan Lee for pointing this out to me.)* Imagine we have *two rulers*, one which goes in the oven with the rod and one that stays cold. The hot rod reaches the 20.11 cm mark on the hot ruler – what does the cold ruler measure it to be? It is simply the length that a 20.11 cm "chunk" of the cold ruler would expand to:

$$L(\text{hot rod, cold ruler}) = (1+\alpha_1 \Delta T)L_2^{hot} = 20.1653 \tag{1.17}$$

This can then be directly compared to the length of the cold rod (on the cold ruler) to get

$$\Delta L_2 = L(\text{hot rod, cold ruler}) - L_2^{cold} = (1+\alpha_1 \Delta T)L_2^{hot} - L_2^{cold} = 20.165 - 20.05 = 0.115 \tag{1.18}$$

And then we can get α_2 as

$$\alpha_2 = \frac{\Delta L_2}{L_2^{cold}\Delta T} = \frac{0.115}{20.05 \times 250 \text{ K}} = 23 \times 10^{-6} \text{ K} \tag{1.19}$$

which agrees. (You can check that all the algebra agrees too, not just the numerical value.)

28. A composite bar of length $L = L_1 + L_2$ is made from a bar of material 1 and length L_1 attached to a bar of material 2 and length L_2 as shown in Fig. 21-15.

(a) Show that the effective coefficient of linear expansion α for this bar is given by $\alpha = (\alpha_1 L_1 + \alpha_2 L_2)/L$.

(b) Using steel and brass, design such a composite bar whose length is 52.4 cm and whose effective coefficient of linear expansion is $13 \times 10^{-6}/°C$.

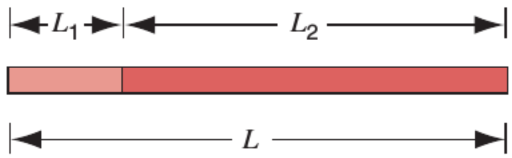

FIGURE 21-15. Exercise 28.

Answer: *(a)* Find the total change in length

$$\Delta L = \Delta L_1 + \Delta L_2 = \alpha_1 L_1 \Delta T + \alpha_2 L_2 \Delta T \tag{1.20}$$

and then put it in the form $\Delta L = \alpha L \Delta T$ as

$$\Delta L = (\alpha_1 L_1 + \alpha_2 L_2)\Delta T = \left(\frac{\alpha_1 L_1 + \alpha_2 L_2}{L}\right) L\Delta T \equiv \alpha L \Delta T \tag{1.21}$$

(b) We are given L and α, and want to solve for L_1, L_2. (We also know α_1, α_2.) Have two unknowns, so need two equations. They are

$$L = L_1 + L_2 \tag{1.22}$$
$$\alpha = (\alpha_1 L_1 + \alpha_2 L_2)/L \tag{1.23}$$

Rearrange the second

$$(\alpha/\alpha_1)L = L_1 + (\alpha_2/\alpha_1)L_2 \tag{1.24}$$

then subtract it from the first to get

$$(1 - \alpha/\alpha_1)L = (1 - \alpha_2/\alpha_1)L_2 \implies L_2 = \frac{1 - \alpha/\alpha_1}{1 - \alpha_2/\alpha_1} L = \frac{\alpha_1 - \alpha}{\alpha_1 - \alpha_2} L \tag{1.25}$$

and then

$$L_1 = L - L_2 = \left(1 - \frac{\alpha_1 - \alpha}{\alpha_1 - \alpha_2}\right) L = \frac{\alpha - \alpha_2}{\alpha_1 - \alpha_2} L \tag{1.26}$$

Use $\alpha_1 = \alpha_{brass} = 19 \times 10^{-6}/K$ and $\alpha_2 = 11 \times 10^{-6}/K$ to get

$$L_1 = \frac{13 - 11}{19 - 11} L = \frac{1}{4} L = 13.1 \text{ cm}, \qquad L_2 = \frac{19 - 13}{19 - 11} L = \frac{3}{4} L = 39.3 \text{ cm} \tag{1.27}$$

34. An aluminum cup of 110 cm^3 capacity is filled with glycerin at 22 °C. How much glycerin, if any, will spill out of the cup if the temperature of the cup and glycerin is raised to 28 °C? (The coefficient of volume expansion of glycerin is $5.1 \times 10^{-4}/°C$)

Answer: : Use $\beta_{alum} \approx 3\alpha_{alum} = 69 \times 10^{-6}/K$, so

$$\Delta V_{alum.} = \beta_{alum} V_0 \Delta T, \qquad \Delta V_{glyc} = \beta_{glyc} V_0 \Delta T \tag{1.28}$$

Since $\beta_{glyc} > \beta_{alum}$ some amount will spill out:

$$V_{spill} = \Delta V_{glyc} - \Delta V_{alum} = (\beta_{glyc} - \beta_{alum}) V_0 \Delta T = (511 - 69) \times 10^{-6} \times 110 \text{ cm}^3 \times 6 = \boxed{0.29 \text{ cm}^3} \tag{1.29}$$

1.c RHK Problems

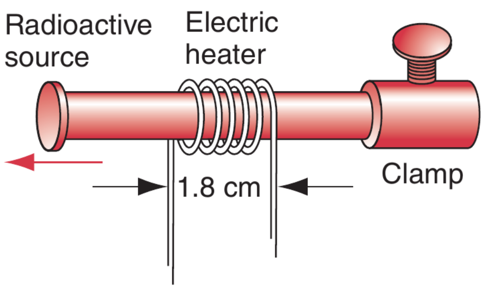

FIGURE 21-18. Problem 6.

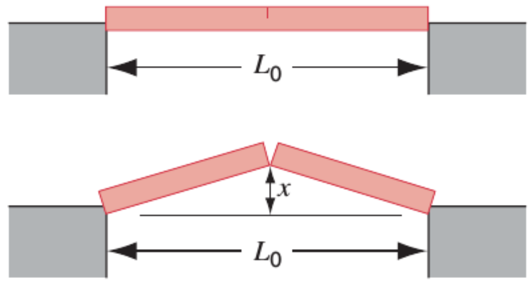

FIGURE 21-19. Problem 8.

6. In a certain experiment, it was necessary to be able to move a small radioactive source at selected, extremely slow speeds. This was accomplished by fastening the source to one end of an aluminum rod and heating the central section of the rod in a controlled way. If the effective heated section of the rod in Fig. 21-18 is 1.8 cm, at what constant rate must the temperature of the rod be made to change if the source is to move at a constant speed of 96 nm/s?

Answer: If the temperature is increased at a constant rate $\Delta T/s$, then the rod gets longer at a rate $\Delta L/s = (\alpha L_0 \Delta T)/s$ and the radioactive source will move at that same velocity. Therefore we just need

$$96 \text{ nm} = \left(23 \times 10^{-6}/\text{K}\right)(1.8\text{cm})\Delta T \implies \Delta T = \frac{96 \times 10^{-9}}{1.8 \times 10^{-2}} \times \frac{1}{23 \times 10^{-6}} \text{ K} = 0.23 \text{ K} \tag{1.30}$$

That is, the temperature should increase at a rate of $\boxed{0.23K/s}$. *(I believe this is a reference to Mossbauer spectroscopy experiments)*

7. *(a)* Show that if the lengths of two rods of different solids are inversely proportional to their respective coefficients of linear expansion at the same initial temperature, the difference in length between them will be constant at all temperatures.

(b) What should be the lengths of a steel and a brass rod at 0 °C so that at all temperatures their difference in length is 0.30 m?

Answer: *(a)* The wording is not the clearest, but it means we should take

$$\frac{L_1}{L_2} = \frac{\alpha_2}{\alpha_1} \tag{1.31}$$

or equivalently, $\alpha_1 L_1 = \alpha_2 L_2$. Denote the lengths at a different temperature by L_1', L_2'. Then

$$L_2' - L_1' = (1 + \alpha_2 \Delta T)L_2 - (1 + \alpha_1 \Delta T)L_1 = (L_2 - L_1) + (\alpha_2 L_2 - \alpha_1 L_1)\Delta T = L_2 - L_1 \tag{1.32}$$

since $\alpha_2 L_2 - \alpha_1 L_1 = 0$.

(b) Some algebra first

$$x \equiv L_2 - L_1 = L_2 - \frac{\alpha_2}{\alpha_1} L_2 = \left(1 - \frac{\alpha_2}{\alpha_1}\right) L_2 \implies L_2 = \frac{x}{1 - (\alpha_2/\alpha_1)} \tag{1.33}$$

Brass expands more than steel, so we should start with more steel at first, i.e. 1=brass, 2=steel. Then

$$\boxed{L_2 = \frac{0.30 \text{ m}}{1 - (11/19)} = 0.72 \text{ m} \implies L_1 = L_2 - x = 0.42 \text{ m}} \tag{1.34}$$

8. As a result of a temperature rise of 32 °C, a bar with a crack at its center buckles upward, as shown in Fig. 21-19. If the fixed distance $L_0 = 3.77$ m and the coefficient linear expansion is 25×10^{-6} °C, find x, the distance to which the center rises.

Answer: Two right triangles are formed, each with base $L_0/2$, height x, and hypotenuse $L'/2 = (1 + \alpha\Delta T)L_0/2$. Use the Pythagorean theorem,

$$\frac{L_0^2}{4} + x^2 = \frac{(L')^2}{4} = \frac{(1+\alpha\Delta T)^2 L_0^2}{4} \tag{1.35}$$

Rearranging,

$$x^2 = \frac{(1+\alpha\Delta T)^2 L_0^2}{4} - \frac{L_0^2}{4} = \frac{L_0^2}{4}\left((1+\alpha\Delta T)^2 - 1\right) = \frac{L_0^2}{4}(2\alpha\Delta T + (\alpha\Delta T)^2) \tag{1.36}$$

so then

$$x = \frac{L_0}{2}\sqrt{2\alpha\Delta T + (\alpha\Delta T)^2} \approx \frac{L_0}{2}\sqrt{2\alpha\Delta T} = \frac{3.77 \text{ m}}{2}\sqrt{2 \times 25 \times 10^{-6} \times 32} = \frac{3.77}{50} \text{ m} = 7.54 \text{ cm} \tag{1.37}$$

(The $(\alpha\Delta T)^2 = (0.008)^2$ term is totally negligible.)

14. *(Modified)* Glass-to-metal seals are an important part of many electronic components, such as light bulbs. One way to make one is to thread metallic wire through molten glass, then let it cool down. For this problem, assume $\alpha_{glass} = 9 \times 10^{-6}$ / K at all temperatures, even when molten. Additionally, take $\alpha_{copper} = 17 \times 10^{-6}$ / K, and $\alpha_{Invar} = 0.7 \times 10^{-6}$ / K.

(a) A copper wire, initially 1.0 mm at 25 °C, is pushed through a layer of molten glass at 1400 °C, and comes to thermal equilibrium with it at that same temperature. What is its new diameter?

(b) When the wire and glass are cooled back down to 25 °C, what is the **area** of the hole in the glass? What is the cross-section area of the copper wire?

Dumet wire consists of a cylindrical core of Invar (nickel-steel alloy) surrounded by a cylindrical sheath of copper. The diameters of the core and of the sheath are chosen so that the wire duplicates the expansion characteristics of glass, allowing for an airtight seal.

(c) Show that the ratio of the Invar core radius to that of the copper sheath should be

$$\frac{r_{Invar}}{r_{copper}} = \sqrt{\frac{\alpha_{copper} - \alpha_{glass}}{\alpha_{copper} - \alpha_{Invar}}} \tag{1.38}$$

(d) What is a typical value for this ratio?

Answer: *(a)* $L' = (1 + \alpha\Delta T)L = (1 + 17 \times 10^{-6} \times 1375)$ mm $= 1.023$ mm

(b) When hot, $A_{hole} = \pi d_{hole}^2/4 = (\pi/4)(1.023 \text{ mm})^2 = 0.82 \text{ mm}^2$. Upon cooling, the hole shrinks according to the glass's thermal expansion coefficient

$$A'_{hole} = (1 + 2\alpha_{glass}\Delta T)A_{hole} = (1 - 2 \times 9 \times 10^{-6} \times 1375)(0.82) \text{ mm}^2 = 0.802 \text{ mm}^2 \tag{1.39}$$

The cross-section area of the copper wire is

$$A_{wire} = \pi(1.0 \text{ mm})^2/4 = 0.785 \text{ mm}^2 \tag{1.40}$$

Because $\alpha_{copper} > \alpha_{glass}$, the wire has shrunk more than the hole, so the seal is no longer airtight.

(c) To get an airtight seal, we need $\Delta A_{hole} = \Delta A_{wire}$, so that if $A_{hole} = A_{wire}$ when hot (or cold), they remain equal at all temperatures. These are each

$$\Delta A_{hole} = 2\alpha_{glass}\Delta T A_{hole} = 2\alpha_{glass}\Delta T(\pi r_{copper}^2) = 2\pi\Delta T(\alpha_{glass} r_{copper}^2) \tag{1.41}$$

$$\Delta A_{wire} = 2\alpha_{copper}\Delta T A_{copper} + 2\alpha_{Invar}\Delta T A_{Invar} \tag{1.42}$$

$$= 2\Delta T \left(\alpha_{copper}(\pi r_{copper}^2 - \pi r_{Invar}^2) + \alpha_{Invar}(\pi r_{Invar}^2)\right) \tag{1.43}$$

$$= 2\pi\Delta T(\alpha_{copper} r_{copper}^2 - \alpha_{copper} r_{Invar}^2 + \alpha_{Invar} r_{Invar}^2) \tag{1.44}$$

It is helpful to draw a picture of the wire and label the various radii and areas. Setting these ΔA equal to each other,

$$\alpha_{glass} r_{copper}^2 = \alpha_{copper} r_{copper}^2 - \alpha_{copper} r_{Invar}^2 + \alpha_{Invar} r_{Invar}^2 \tag{1.45}$$

Collecting the terms by the radius type,

$$(\alpha_{copper} - \alpha_{Invar}) r_{Invar}^2 = (\alpha_{copper} - \alpha_{glass}) r_{copper}^2 \tag{1.46}$$

Dividing r_{copper}^2 over to the left, and the Invar coefficient over to the right,

$$\frac{r_{Invar}^2}{r_{copper}^2} = \frac{\alpha_{copper} - \alpha_{glass}}{\alpha_{copper} - \alpha_{Invar}} \tag{1.47}$$

Taking the sqrt of both sides gives the desired expression.

(d) Finally, just plug in some numbers (canceling the common factors of 10^{-6}/K)

$$\boxed{\frac{r_{Invar}}{r_{copper}} = \sqrt{\frac{17 - 9}{17 - 0.7}} = 0.70} \tag{1.48}$$