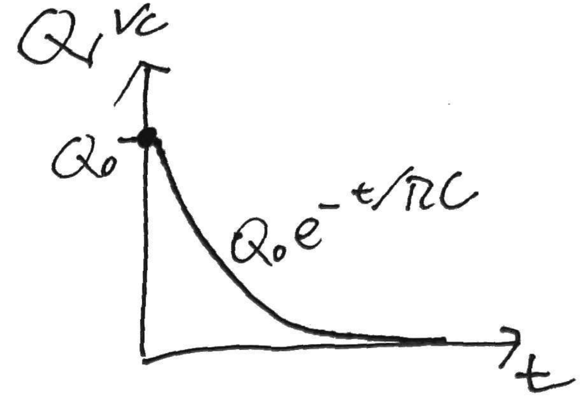
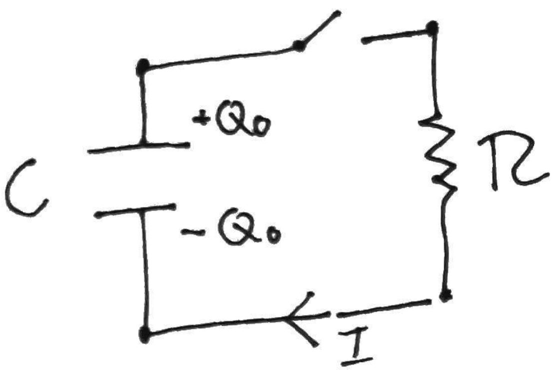
RC Circuits

↳ Non-equilibrium, transient behavior



- Discharging

↳ C initially charged
$Q(t=0)=Q_0$



Close the switch
⇒ Allows current to flow until steady-state is reached.

↳ $V_C = 0$
⇒ $Q = 0$
$(t=\infty)$

→ $\frac{dQ}{dt} = 0$

- Charge cons.

$$\boxed{I = -\frac{dQ}{dt}}$$

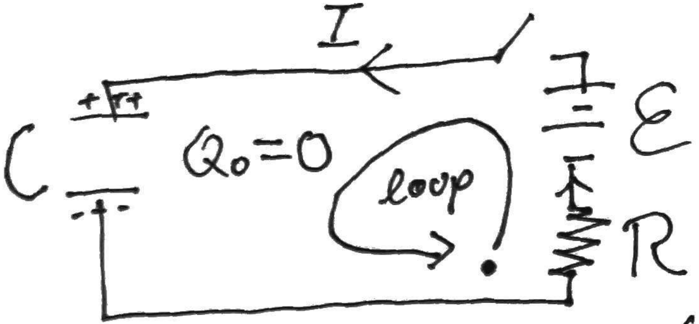
- Loop rule: $V_C = V_R$

↳ $$\frac{Q(t)}{C} = R \cdot I(t) = -R\frac{dQ}{dt}$$

$$\Rightarrow \frac{dQ}{dt} = -\frac{Q(t)}{RC} \Rightarrow \boxed{Q(t) = Q_0 e^{-t/RC}}$$

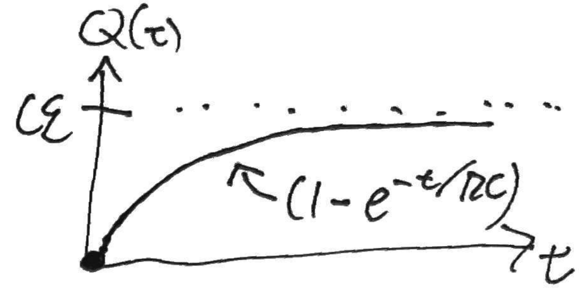
$$\frac{d}{dt}\left[Q(t)\right] = Q_0\left(\frac{-1}{RC}e^{-t/RC}\right)$$
$$= -\frac{Q(t)}{RC} \checkmark$$

• Charging



C initially uncharged

$Q(t=0)=0$



Close the switch
$\Rightarrow$ Current to flow

Charge cons.

$$\boxed{I=+\frac{dQ}{dt}}$$

Steady state: $t=\infty$, $Q=Q_\infty$, $I=+\frac{dQ}{dt} = 0$

$$V_R = IR = 0$$

$$\Rightarrow V_C(t=\infty)=\mathcal{E} \Rightarrow Q_\infty = C\cdot V_C = C\mathcal{E}$$

- Loop rule $(-V_C - V_R + \mathcal{E} = 0)$

$$-I(t)R + \mathcal{E} - Q(t)/C = 0$$

$$\mathcal{E} = \frac{Q(t)}{C} + \frac{dQ}{dt}R$$

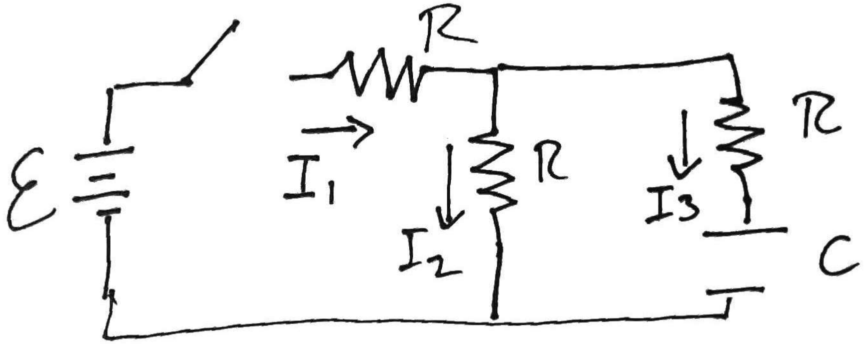
$$\hookrightarrow \boxed{\frac{dQ}{dt} + \frac{Q(t)}{RC} = \mathcal{E}/R}$$

$$\boxed{Q(t) = (1-e^{-t/RC})\underbrace{C\mathcal{E}}_{Q_\infty}}$$

$$\left(\frac{d}{dt}[\ \ ] = +\frac{C\mathcal{E}}{RC}e^{-t/RC}\right.$$

$$\left.\frac{Q(t)}{RC} + \frac{dQ}{dt} = \frac{C\mathcal{E}}{RC} = \mathcal{E}/R \ \checkmark\right)$$

1] (Giancoli 26.49)



$t=0$, C is uncharged

⇒ Close the switch

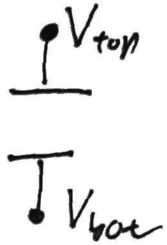
↳ Look at $t=0$, $t=\infty$ behavior.

Often we can consider a simpler, effective circuit.

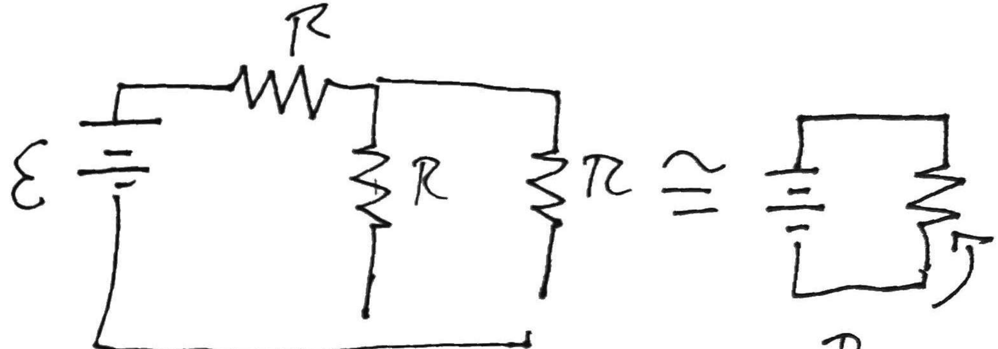
(a) At $t=0$, what are the currents?

↳ $Q(0)=0$

$$\Rightarrow \underbrace{V_c(0) = Q/C = 0}$$



$V_{top} = V_{bot}$ ⇒ I can replace C with wire.



Parallel → $R/2$

$R_{eff} = 3R/2$

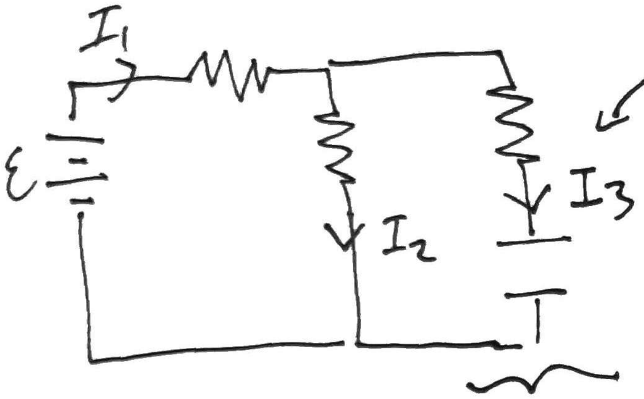
$$\Rightarrow I_1 = \frac{\varepsilon}{R_{eff}} = \frac{2\varepsilon}{3R},$$

$I_2 = I_3$ (sym.) $= \varepsilon/3R$ (Junc. rule)

(b) Currents at $t=\infty$?

Q_c has reached a constant Q_∞

$\Rightarrow \dfrac{dQ_c}{dt} = 0$

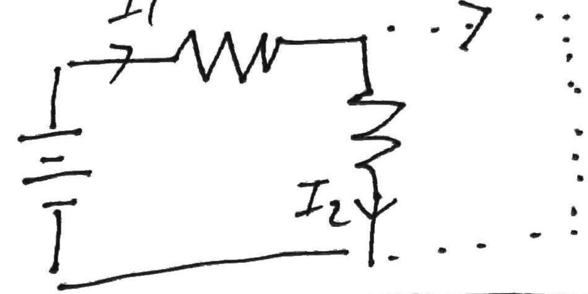


$\dfrac{dQ}{dt} = I_3$

$\boxed{I_3 = \dfrac{dQ}{dt} = 0}$ at $t=\infty$

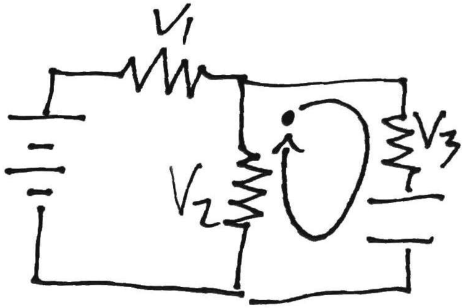
$\Rightarrow$ Remove the right branch from circuit.

$\rightarrow$



$\Rightarrow \boxed{I_1 = I_2 = \dfrac{\varepsilon}{R_{eff}} = \dfrac{\varepsilon}{2R}}$

(c) At $t=\infty$, what is V_c, Q?



Loop rule:

$V_2 = V_3 + V_c$

$\underbrace{I_2 R}_{\frac{\varepsilon}{2R}\cdot R} = \underbrace{I_3}_{0} R + V_c$

$\Rightarrow \boxed{V_c = \varepsilon/2}$.

$\boxed{Q_\infty = CV_c = C\varepsilon/2}$

(b) $t=\infty$?

Q_C has reached a constant val.

↳ $\frac{dQ_c}{dt}=0$

$\quad = I_3$

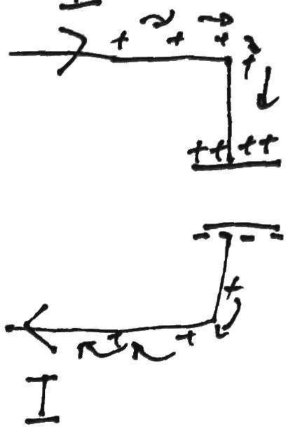
↳

$$I_1 = I_2 = \frac{\varepsilon}{2R}$$

(c) $t=\infty$, what is V_c?

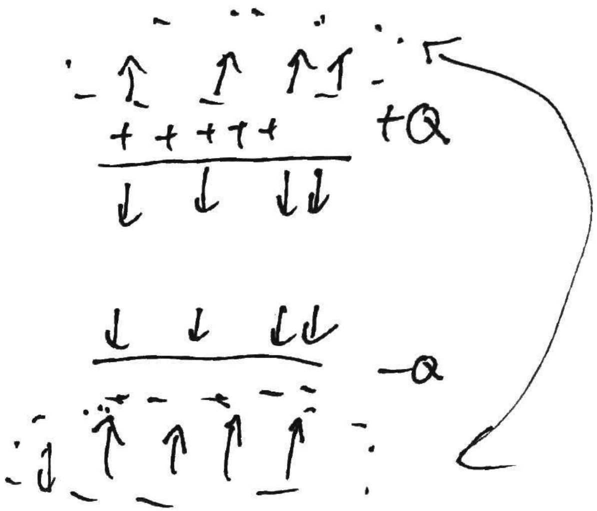
↳

$V = I_2 R = \varepsilon/2$



← + charge gets stuck / builds up

+ charge pulled off of negative plate.

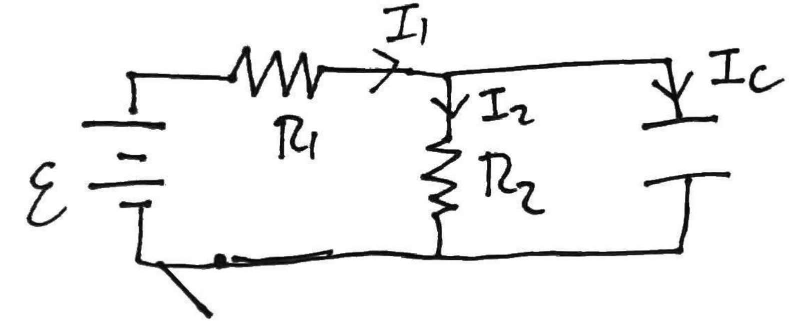


Unless Q on top and bot plates are equal,

⇒ net E-field.

↳ Will push or pull charge until equalized.

2) (Giancolli 26.50)



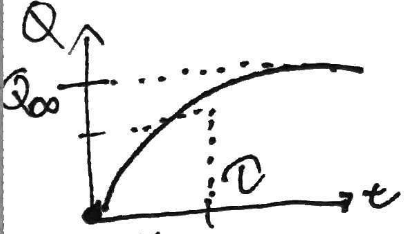
Initially, C uncharged.

↳ Close switch at $t=0$.

<u>Charging process</u>

↳ I expect something like

$$Q(t) = \left(1 - e^{-t/\tau}\right) \cdot Q_\infty$$



<u>Time constant</u>

"ansatz" (guess)

(a) Determine τ (time const.)

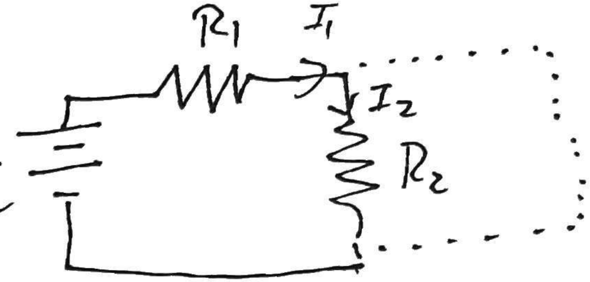
(b) Find Q_∞

↳ $t=\infty$, $Q = Q_\infty$ const

$$\frac{dQ}{dt} = 0 \quad (t \to \infty)$$

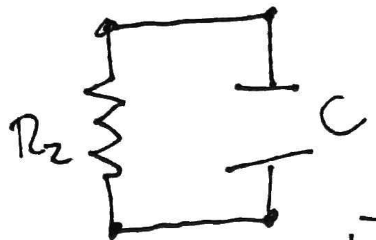
↳ $I_c = 0$

Simpler eff. circuit:



$$I_1 = I_2 = \frac{\mathcal{E}}{R_{eff}} = \frac{\mathcal{E}}{R_1 + R_2}$$

Loop rule:



$$V_2 = V_c$$

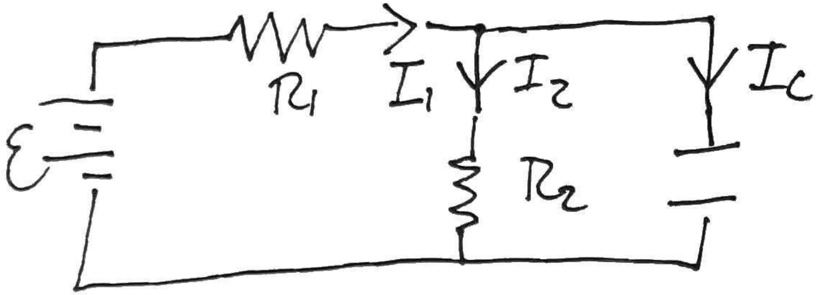
$$I_2 R_2 = Q_\infty / C$$

$$\boxed{Q_\infty = \mathcal{E} \cdot \frac{R_2}{R_1 + R_2} \cdot C}$$

(✓ units work)

(a) (i) Use Kirchoff laws

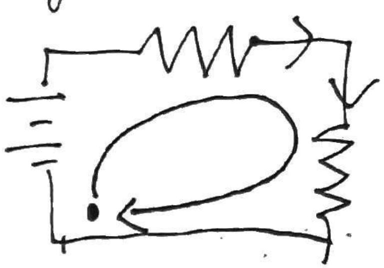
(analyze circuit from scratch)



Junction rule: $\boxed{I_1 = I_2 + I_C}$

Loop rule:

Left loop -



$$0 = +\mathcal{E} - I_1 R_1 - I_2 R_2$$

$$\boxed{\mathcal{E} = I_1 R_1 + I_2 R_2}$$

Right loop

$$\Rightarrow V_c = I_2 R_2$$

$$\boxed{\frac{Q}{C} = I_2 R_2}$$

(ii) ☆ RC circuit strategy ☆

☆ ☆ ☆

Rewrite all the I in terms of Q, $\frac{dQ}{dt}$.

$$\boxed{I_c = \frac{dQ}{dt}}$$

$$\boxed{I_2 = \frac{Q}{R_2 C}}$$

$$\boxed{I_1 = \frac{Q}{R_2 C} + \frac{dQ}{dt}}$$

(iii) Use leftover equation to get $Q, \frac{dQ}{dt}$ relationship (ODE).

$$\boxed{\mathcal{E} = R_1\left[\frac{Q}{R_2 C} + \frac{dQ}{dt}\right] + \frac{Q}{C}}$$

$\mathcal{E}$, R_1, R_2, C are constants

(iv) Try to solve:

$$\mathcal{E} = R_1 \left[\frac{Q}{R_2 C} + \frac{dQ}{dt} \right] + \frac{Q}{C}$$

$$\frac{\mathcal{E}}{R_1} = \frac{Q}{R_2 C} + \frac{Q}{R_1 C} + \frac{dQ}{dt}$$

$$= \frac{Q}{C} \underbrace{\left(\frac{1}{R_1} + \frac{1}{R_2} \right)} + \frac{dQ}{dt}$$

$$= R_{12}^{-1} \equiv \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$ (Parallel formula for R)

$$\boxed{\mathcal{E}/R_1 = \frac{Q}{R_{12} C} + \frac{dQ}{dt}}$$

↳ Solve

↳ or, try the ansatz: $Q(t) = Q_\infty \left(1 - e^{-t/\tau}\right)$

(earlier $Q_\infty = C\mathcal{E} \cdot \frac{R_2}{R_1 + R_2}$)

$$\frac{\mathcal{E}}{R_1} = \frac{Q_\infty}{R_{12} C}\left(1 - e^{-t/\tau}\right) + \frac{Q_\infty}{\tau} e^{-t/\tau}$$

Rewrite:
(Divide by Q_∞, move const. over)

$$\boxed{\left(\frac{\mathcal{E}}{R_1 Q_\infty} - \frac{1}{R_{12} C} \right) = \left(\cancel{Q}\frac{1}{\tau} - \frac{1}{R_{12} C} \right) e^{-t/\tau}}$$

LHS: $$\frac{\mathcal{E}}{R_1 Q_\infty} - \frac{1}{R_{12} C} = \frac{\mathcal{E}}{R_1}\left(\frac{1}{C\mathcal{E}}\,\frac{R_1+R_2}{R_2}\right) - \frac{1}{R_{12}C}$$

$$= \frac{R_1+R_2}{C R_1 R_2} - \frac{1}{R_{12}C}$$

$$= \frac{1}{R_{12}C} - \frac{1}{R_{12}C} = 0$$

$$\Rightarrow \underset{\left(\frac{1}{\tau} - \frac{1}{R_{12}C}\right) e^{-t/\tau}}{\underset{\shortparallel}{RHS}} = 0 \qquad \Rightarrow \boxed{\boxed{\tau = R_{12}\, C}}$$

$$R_{12} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$