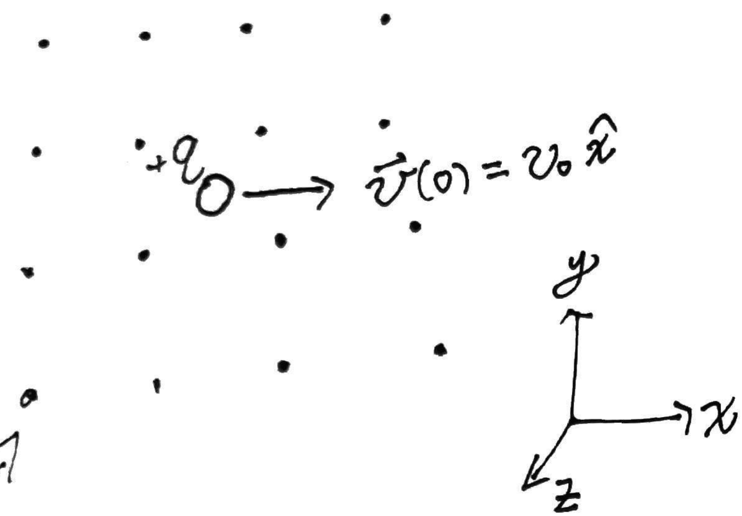
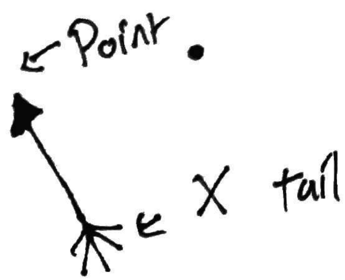
1) "Cyclotron motion"

Circular motion of charged particle in uniform B-field.

$(x, y, z)(t=0) = 0$



(Uniform B-field pointing out of the page. $(+\hat{z})$ direction



)

(a) Force at $t=0$?

$$\vec{F} = q\,(\underset{\underset{\displaystyle v_0 \hat{x}}{\downarrow}}{\vec{v}} \times \underset{\underset{\displaystyle B\hat{z}}{\downarrow}}{\vec{B}})$$

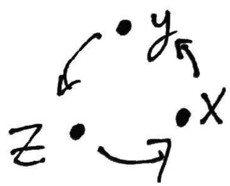
$$= q\, v_0 B_0 (\hat{x} \times \hat{z})$$

$$= q\, v_0 B_0 \,(-\hat{y})$$

$\vec{v}$, $\Downarrow F_y$

(Reminder:

$\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\underbrace{\hat{z} \times \hat{x} = \hat{y}}_{\text{Tricky one}}$



(If $\vec{a} \times \vec{b} = \vec{c}$
$\Rightarrow \vec{b} \times \vec{a} = -\vec{c}$) $\Rightarrow$ $-\vec{a} \times \vec{a} = \vec{a} \times \vec{a} = 0$

$\hat{x} \times \hat{z} = -\hat{z} \times \hat{x} = -\hat{y}$.)

(b) Suppose at t, $\vec{v}(t)$.
What is $\vec{F}(t)$?

$$\vec{F} = q\,\cancel{|\vec{v}|\,B\,(\hat{v}\times B}$$
$$= qB\,(\vec{v}\times\hat{z})$$

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

$$\vec{v}\times\hat{z} = v_x\underbrace{\hat{x}\times\hat{z}}_{-\hat{y}} + v_y\underbrace{\hat{y}\times\hat{z}}_{\hat{x}} + v_z\underbrace{\hat{z}\times\hat{z}}_{0}$$

$$= -v_x\hat{y} + v_y\hat{x}$$

$$\Rightarrow \vec{F} = (qBv_y)\hat{x} - (qBv_x)\hat{y}$$

$F_z = 0$ ← More generally, $\vec{F}\perp\vec{B}$

(also $\vec{F}\perp\vec{v}$)

(c) Because $\vec{F}\perp\vec{v}$

↪ No work is done

$$W = \vec{F}\cdot\vec{v} = 0$$

Total kinetic energy is const.

$\Vert$

$mv^2 \Rightarrow \boxed{|\vec{v}| = \text{const.}}$

$F_z = 0$, ($\cancel{v_z(0)=0}$)

$\Rightarrow v_z$ is const ~~≠0~~.

F=ma

$\Rightarrow a_z = 0$

Initially, $v_z = 0$

$\Rightarrow v_z = 0$ ($\forall t$) always

(Aside:

Get a spiral shape if $v_z(0)\neq 0$)

(d) Apply Newton's ~~3rd~~ 2nd Law:

$$\vec{f}(\vec{v}) = \vec{F} = m\vec{a} = m\frac{d}{dt}\vec{v}$$

(Know how to get $\vec{F}$, $\vec{a}$ in terms of $\vec{v}$)

$$qBv_y = F_x = m\frac{dv_x}{dt}$$

$$-qBv_x = F_y = m\frac{dv_y}{dt}$$

$$\left(0 = F_z = m\frac{dv_z}{dt}\right)$$

$$\left.\boxed{\begin{aligned}\frac{dv_x}{dt} &= \left(\frac{qB}{m}\right)v_y \\ \frac{dv_y}{dt} &= \left(-\frac{qB}{m}\right)v_x\end{aligned}}\right\} \text{Coupled ODE}$$

Notice: To relate v_y back to v_x, just have to take derivative:

$$\frac{d}{dt}\left[\frac{dv_x}{dt}\right] = \frac{qB}{m}\frac{dv_y}{dt}$$

$$\frac{d^2}{dt^2}v_x = -\left(\frac{qB}{m}\right)^2 \cancel{\frac{dv_y}{dt}}\, v_x$$

(Compare: $\ddot{x} = -\omega^2 x$ harmonic motion.)

$$\Rightarrow \boxed{\begin{aligned}&v_x = A\cdot\cos(\omega t)\\ &\omega = qB/m, \quad A = v_x(0)\end{aligned}}$$

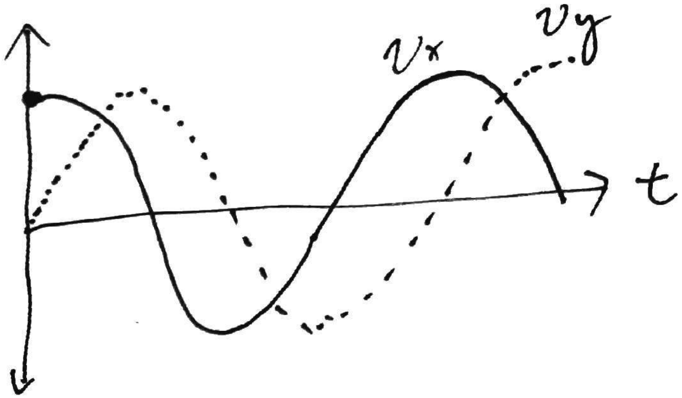
$$v_y = \left(\frac{m}{qB}\right)\frac{dv_x}{dt}$$

$$= \frac{m}{qB}\frac{d}{dt}\left[v_0 \cos(\omega t)\right]$$

$$= \underbrace{\frac{m}{qB}}_{= 1/\omega}\cdot \cancel{\omega}\left[-\omega v_0 \sin(\omega t)\right]$$

$$\boxed{v_y = -v_0 \sin(\omega t)} \qquad (v_z = 0)$$

$$\boxed{v_x = v_0 \cos(\omega t)}$$



$$\omega = \frac{qB}{m}$$

(e) Integrate to get position

$$x = \int_0^t v_x \, dt + \underbrace{x(0)}_{= 0}$$

$$= \frac{v_0}{\omega} \sin(\omega t)$$

$$\left(y = \frac{v_0}{\omega}\left[\cos(\omega t) - 1\right]\right)$$

$$y = \int_0^t v_y(t')\, dt' + y(0)^{=0}$$

$$= \left[\frac{v_0}{\omega}\cos(\omega t')\right]_0^t = \frac{v_0}{\omega}\left[\cos \omega t - 1\right]$$

$$\frac{dv_x}{dt} = \frac{qB}{m} v_y$$

$$\frac{d}{dt}\left[\frac{dv_x}{dt}\right] = \frac{d}{dt}\left[\frac{qB}{m} v_y\right]$$

$$\frac{d^2 v_x}{dt^2} = \frac{qB}{m}\frac{dv_y}{dt}$$

$\leftarrow$ use $\frac{dv_y}{dt} = -\frac{qB}{m} v_x$

$$= \frac{qB}{m}\left(-\frac{qB}{m} v_x\right)$$

$$\frac{d^2}{dt^2} v_x = \underbrace{-\left(\frac{qB}{m}\right)^2}_{\omega^2} v_x$$

Most gen sol:

$v_x(t) = A\cos(\omega t + \phi)$

($\phi = \pi/2$ $\rightarrow$ sin)

$$\frac{d^2}{dt^2} v_x(t) = -\omega^2 v_x(t)$$

$\rightarrow$ v_x could be cos, or sin

$\rightarrow$ $v_x = v$ to start with $\rightarrow$ pick cos.

$\underset{0}{\overset{\neq}{}}$

$$\Rightarrow \boxed{v_x(t) = \underset{\substack{\| \\ v_0}}{v_x(0)} \cdot \cos(\omega t)}$$

(f) $x(t) = \dfrac{v_0}{\omega} \sin \omega t$

$y(t) = \dfrac{v_0}{\omega} [\cos \omega t - 1]$

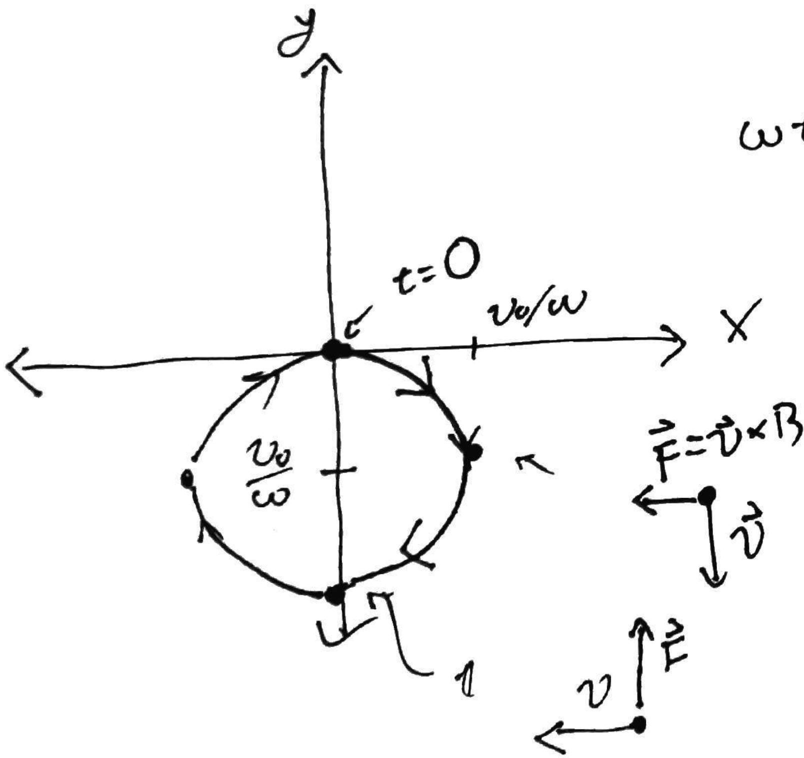
$$\boxed{\omega = \frac{qB}{m}}$$

"Cyclotron frequency"

$\Rightarrow$ Uniform Circular motion with radius

$$\boxed{r = \frac{v_0}{\omega} = \frac{v_0 m}{qB}}$$

$\hookrightarrow q, B$ large $\Rightarrow$ Fast, tight loops.



$\omega t = \pi/2 \rightarrow x = \dfrac{v_0}{\omega}$

$y = \dfrac{v_0}{\omega} [\overset{\nearrow 0}{\cos \tfrac{\pi}{2}} - 1]$

$= \dfrac{v_0}{\omega}(-1)$

v_0 large $\hookrightarrow$ Larger loops.

$\omega t = \pi \rightarrow x = 0$

$y = -2 \dfrac{v_0}{\omega}$

★ Lesson:

For motion in magnetic field, solve for $\underline{\underline{\vec{v} \text{ first}}}$.