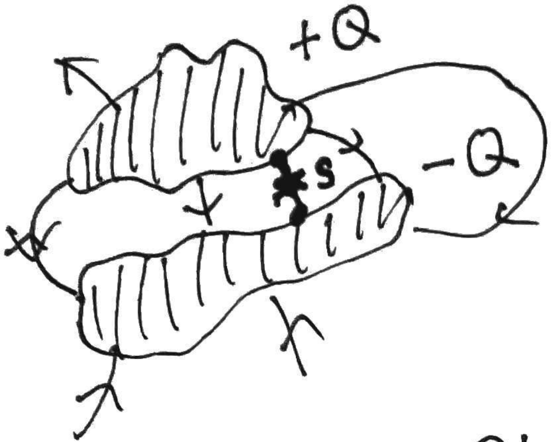
Capacitance



$$\Delta V = -\int_{-}^{+} \vec{E} \cdot d\vec{s}$$

$$\Delta V = \frac{1}{C} Q$$

1. Place charges $\pm Q$ on plates
2. Find resulting $\vec{E}$
   - Gauss' Law
   - Superposition
3. Determine $\Delta V = -\int_{-}^{+} \vec{E} \cdot d\vec{s}$
4. Calculate $C = \dfrac{Q}{\Delta V}$

$$\Rightarrow \quad C = \varepsilon_0 \cdot \frac{[\text{Area}]}{[\text{Distance}]} \cong \varepsilon_0 \cdot [\text{Length}]$$

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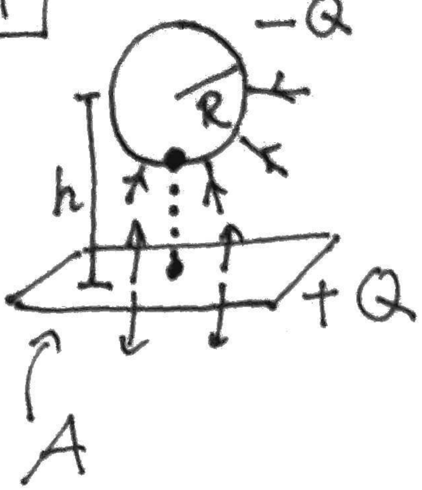
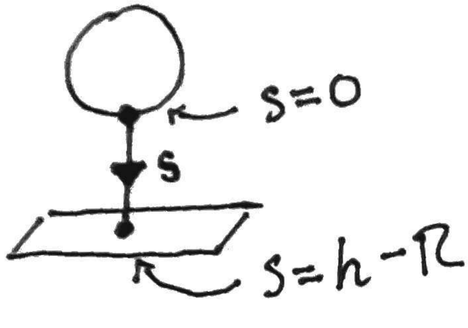


$$\vec{E} = \vec{E}_{sphere} + \vec{E}_{sheet}$$

$$\vec{E}_{sphere} = \frac{1}{4\pi\varepsilon_0}\frac{-Q}{r^2}\hat{r}$$

to center

$$\vec{E}_{sheet} = \frac{\sigma}{2\varepsilon_0}(\pm\hat{z}) \quad , \sigma = Q/A$$



$$\vec{E}_{sphere} = \frac{1}{4\pi\varepsilon_0}\frac{-Q}{(R+s)^2}\hat{s}$$

$$\vec{E}_{sheet} = \frac{\sigma}{2\varepsilon_0}(-\hat{s})$$

Potential/Voltage Difference: $\Delta V = -\int_-^+ \underline{\vec{E}\cdot d\vec{s}}$

Q/A

$$= -\int_0^{h-R} ds \left[\frac{1}{4\pi\varepsilon_0}\frac{-Q}{(R+s)^2} - \frac{\sigma}{2\varepsilon_0}\right]$$

$$\left(\int \frac{1}{(a+x)^2}dx = -\frac{1}{x+a}\right)$$

$$= \frac{Q}{2\varepsilon_0}\int_0^{h-R} ds\left[\frac{1}{2\pi}\frac{1}{(R+s)^2} + \frac{1}{A}\right]$$

$$= \frac{Q}{2\varepsilon_0}\left[\left(\frac{1}{2\pi}\frac{1}{(R+0)} - \frac{1}{2\pi}\frac{1}{(R+h-R)}\right) + \frac{h-R}{A}\right]$$

$$= \frac{Q}{2\varepsilon_0}\left[\frac{1}{2\pi}\frac{h-R}{hR} + \frac{h-R}{A}\right] = \frac{Q(h-R)}{2\varepsilon_0}\left[\frac{1}{2\pi hR} + 1/A\right]$$

$$C = \frac{Q}{\Delta V} = \boxed{\frac{2\varepsilon_0}{(h-R)}\left[\frac{1}{2\pi h R} + \frac{1}{A}\right]^{-1}}$$

(b) $$\lim_{A\to\infty} C = \frac{2\varepsilon_0}{(h-R)}\left[\frac{1}{2\pi h R} + 0\right]^{-1}$$

$$= \cancel{2}\ \varepsilon_0 \cdot \frac{4\pi h R}{(h-R)} \approx \varepsilon_0 \cdot \frac{4\pi R^2}{d}$$

$h \approx R + d$

(c) Stored energy (energy density in E-field: $u = \frac{1}{2}\varepsilon_0 E^2$)

$$U = \frac{1}{2} C V^2 = \frac{Q^2}{2C} = \frac{1}{2} Q V$$

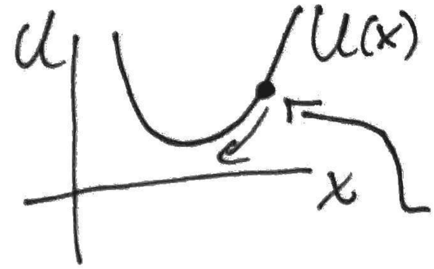
$$U = \frac{1}{2} Q \cdot \frac{Q(h-R)}{2\varepsilon_0}\left[\frac{1}{2\pi R h} + \frac{1}{A}\right]$$

$$= \frac{Q^2}{4\varepsilon_0}(h-R)\left[\frac{1}{2\pi R h} + \frac{1}{A}\right]$$

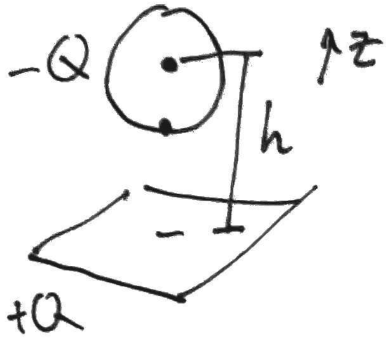
$$\lim_{A\to\infty} U = \frac{Q^2}{8\pi\varepsilon_0}\,\frac{h-R}{Rh} = \frac{Q^2}{8\pi\varepsilon_0 R}\left(1 - R/h\right)$$

$$-\frac{dU}{dh} = \frac{-Q^2}{R\cdot 8\pi\varepsilon_0}\left(+\frac{R}{h^2}\right)$$

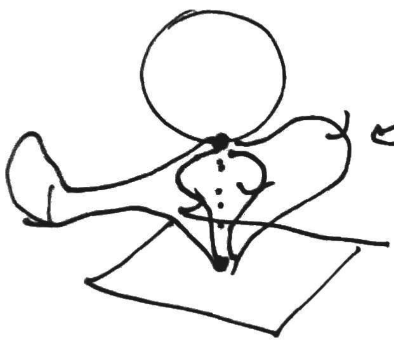
$$= \frac{-Q^2}{8\pi h^2\varepsilon_0} = \frac{-1}{2}\,\frac{1}{4\pi\varepsilon_0}\,\frac{Q^2}{h^2}$$



$$F_x = -\frac{dU}{dx}$$



$$F_z = -\frac{dU}{dh} = -\frac{1}{2}\,\frac{1}{4\pi\varepsilon_0}\,\frac{Q^2}{h^2}$$



Along any of these, same $\Delta V = -\int_{-}^{+} \vec{E}\cdot d\vec{s}$

Not the same, because sphere & sheet are not equipotential surfaces.