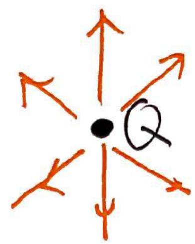
Ampere's Law

- Overview
- Thick wire
- Wire with cavity
- Solenoid + Toroid

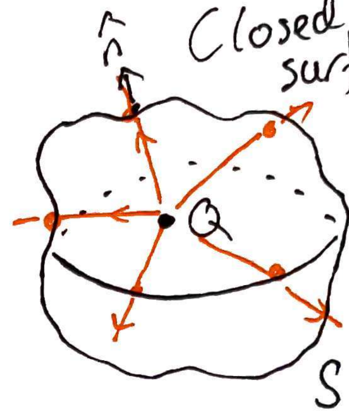
} Symmetric current configurations

(Infinite plane or slab of current)

(Electric) Gauss' Law: Charges create $\vec{E}$-fields

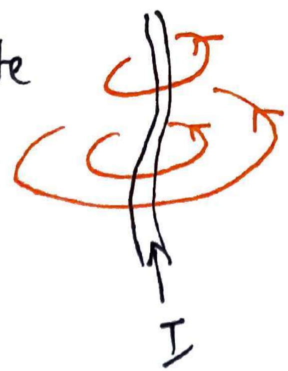


Closed surface, S

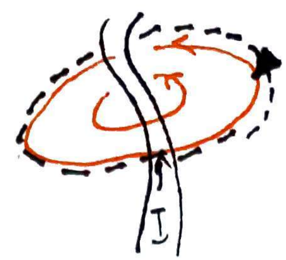


$$\Phi \equiv \int_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

Ampere's Law: Currents create $\vec{B}$-fields.

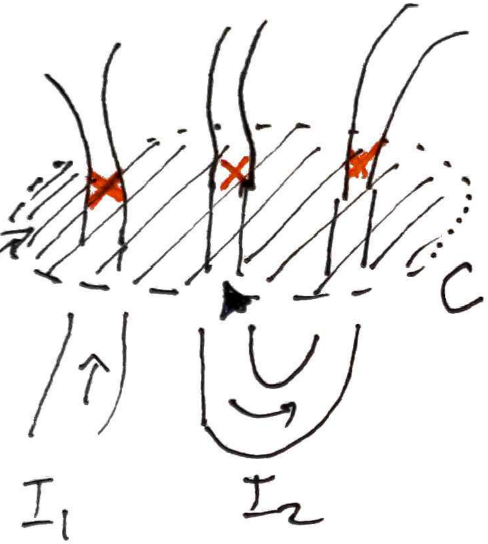


Closed loop, C

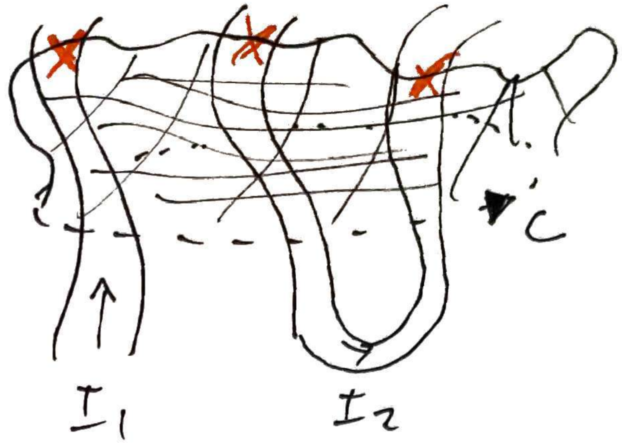


$$\int_C \vec{B} \cdot d\vec{l} = I_{enc} \cdot \mu_0$$

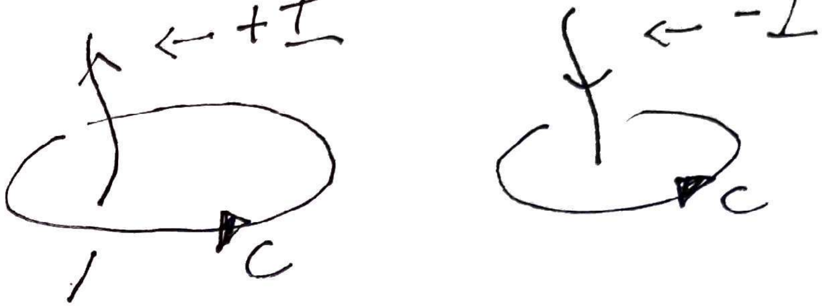
I_{enc}?



Some surface with C as its boundary.



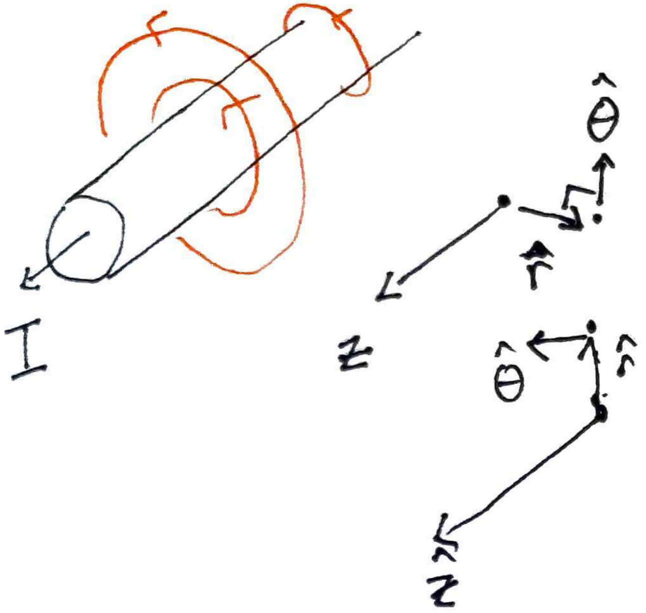
Rule for direction + sign:



$$I_{enc} = I_1 - I_2 + I_2 = I_1$$

1) Long, straight wire

Radius a



Symmetries: Cylindrical

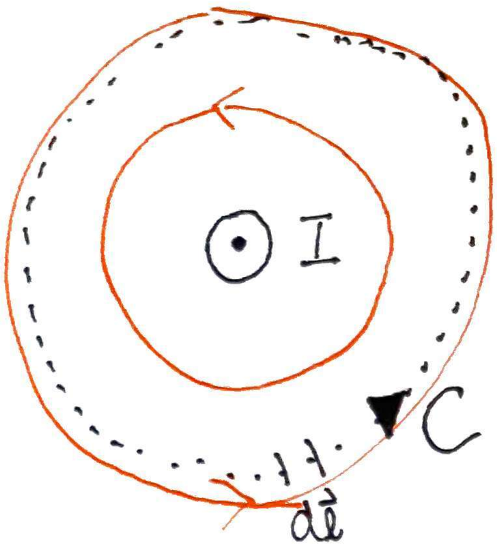
- Translation (z-axis)
  $\hookrightarrow \vec{B}(\vec{r}) = \vec{B}(r, \theta)$
- Rotational
  $\hookrightarrow |B(\vec{r})| = B(r)$
- Reflectional
  $\hookrightarrow \boxed{\vec{B}(\vec{r}) = B(r)\,\hat{\theta}}$

$$\boxed{\hat{\theta} = \hat{z} \times \hat{r}}$$

Ampere's Law: $\oint_C \underbrace{\vec{B} \cdot d\vec{\ell}} = \mu_0 I_{enc}$

Typically, go along or perp. to B-field. $\rightarrow$ Want to choose C to make $\vec{B} \cdot d\vec{\ell}$ as simple as possible (ideally constant).

Outside the wire



C : circle of radius r

$\hookrightarrow d\vec{l} = \underline{r\,d\theta}\,\hat{\theta}$

$$\vec{B}(\vec{r}) = B(r)\,\hat{\theta}$$

$$\vec{B}\cdot d\vec{l} = (B(r)\,\hat{\theta})(r\,d\theta\,\hat{\theta})$$

$$= B(r)\,r\,d\theta\,\underbrace{\hat{\theta}\cdot\hat{\theta}}_{1}$$

LHS: $\displaystyle\oint_{C(r)} \vec{B}\cdot d\vec{l} = \int_0^{2\pi} B(r)\,r\,d\theta = B(r)\,r\cdot 2\pi$

RHS: $\mu_0 I_{enc} = \mu_0 I$

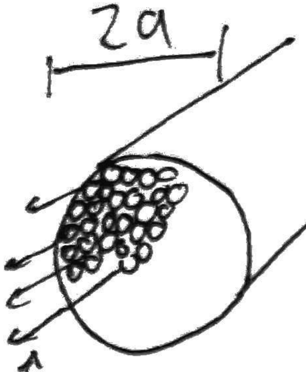
True for *any* cylindrical conf.

$$\Rightarrow B(r)\,2\pi r = \mu_0 I_{enc}^{(r)} \Rightarrow \boxed{B(r) = \frac{\mu_0 I_{enc}(r)}{2\pi r}}$$

$$\Rightarrow \boxed{\vec{B}(\vec{r})_{\text{outside}} = \frac{\mu_0 I}{2\pi r}\,\hat{\theta}}$$

Inside the wire

↪ Uniform current density



"Bundle of wires"

i through small wires

$$\Rightarrow I = \sum i = \int dA \; i(r,\theta)$$

$$\text{Uniform} \Rightarrow i(r,\theta) = i$$

$$\Rightarrow I = i \cdot \pi a^2$$

$$\hookrightarrow i = \frac{I}{A} = \frac{I}{\pi a^2}$$

v const. velocity

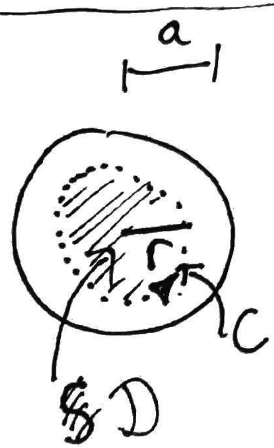
Uniform charge density ρ

$$\Rightarrow i = \rho v$$

$$(I = i \cdot A = \rho v A)$$

(Check: Units?

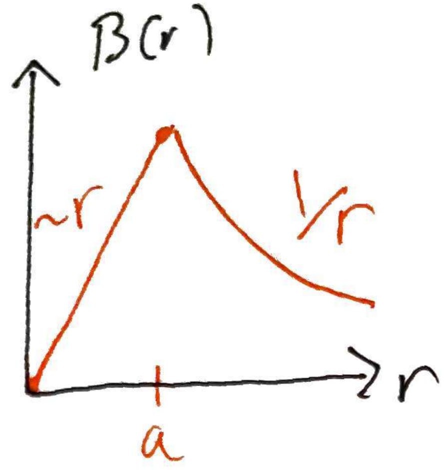
$$[\rho v] = \frac{\text{charge}}{\text{meter}^3} \cdot \frac{\text{meter}}{\text{second}} = \frac{\text{charge}}{\text{second}} \; \frac{1}{\text{meter}^2} = \frac{[I]}{[A]}$$



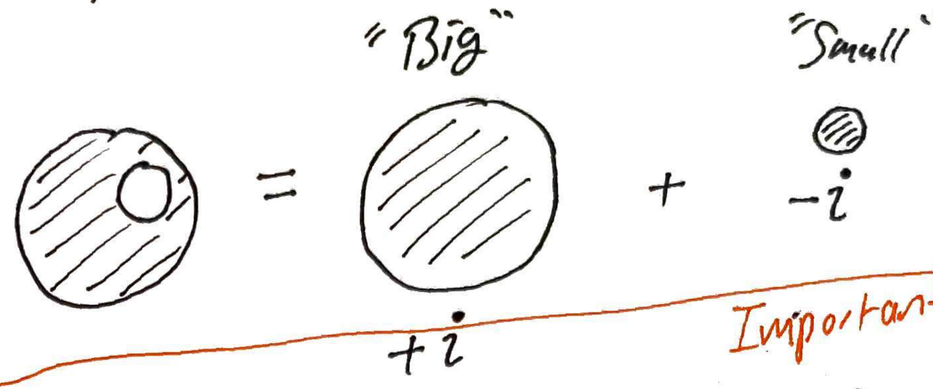
$$I_{enc}^{(r)} = \int_{\oint D} i \, dA = i \cdot A_{enc} = i \cdot \pi r^2 = I \cdot (r/a)^2$$

$$B(r) = \frac{\mu_0 I}{2\pi} \left(\frac{r}{a}\right)^2 \cdot \frac{1}{r} = \boxed{\frac{\mu_0 I}{2\pi} \frac{r}{a^2}} = \frac{\mu_0 i}{2} r$$

Plot :

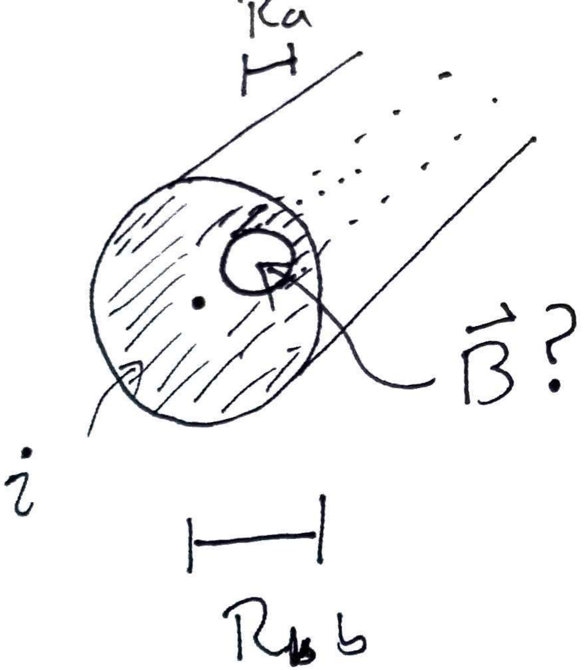


Superposition:



Important piece.

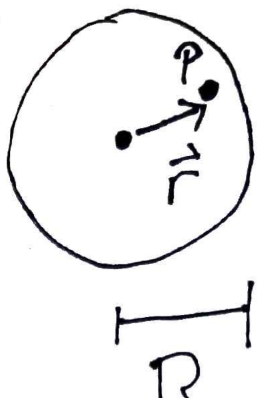
★ | Wire with cavity



$\hookrightarrow$ Everywhere, $\vec{B} = \vec{B}_{big} + \vec{B}_{small}$

~~We know that~~ Cavity : Inside both big and small.

Found before :

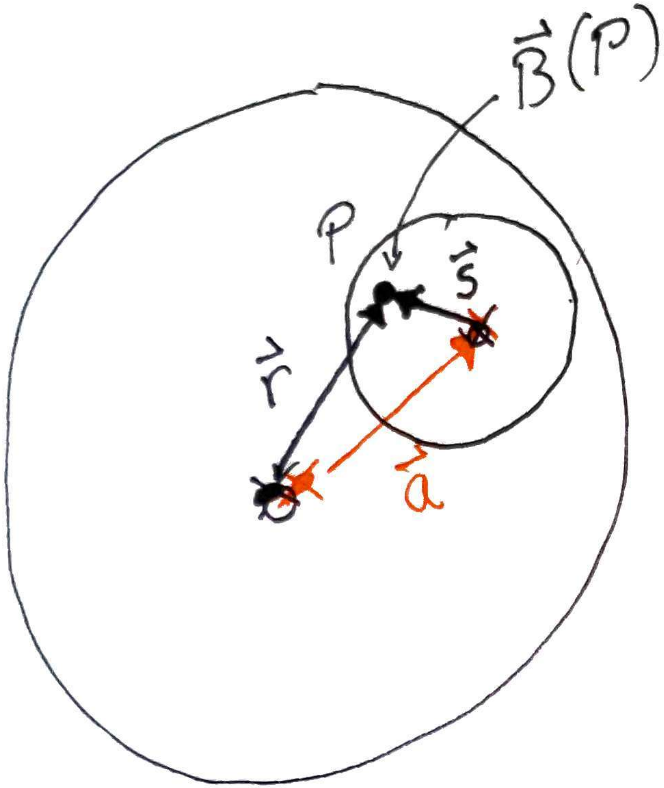


$$\vec{B}(P) = \frac{\mu_0 i}{2} r \hat{\theta}$$

★ Trick $\hat{\theta} = \hat{z} \times \hat{r}$

$$\vec{B}(P) = \frac{\mu_0 i}{2} r (\hat{z} \times \hat{r})$$

$$\boxed{\vec{B}(P) = \frac{\mu_0 i}{2} (\hat{z} \times \vec{r})}$$



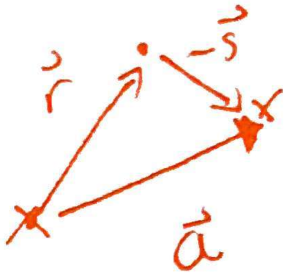
$$\vec{B}_{big} = \frac{\mu_0 i}{2}(\hat{z} \times \vec{r})$$

$$\vec{B}_{small} = -\frac{\mu_0 i}{2}(\hat{z} \times \vec{s})$$

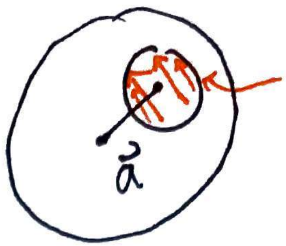
$$\vec{B} = \vec{B}_{big} + \vec{B}_{small}$$

$$= \frac{\mu_0 i}{2}\left[(\hat{z} \times \vec{r}) - (\hat{z} \times \vec{s})\right]$$

$$= \frac{\mu_0 i}{2}\,\hat{z} \times \left[\vec{r} - \vec{s}\right]$$ * Trick 2

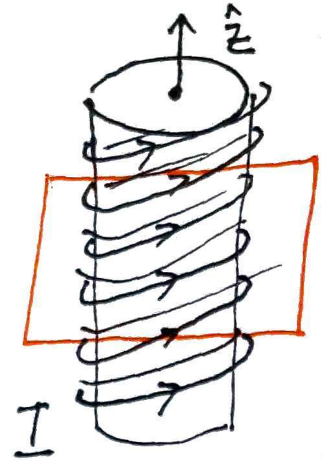


$$\boxed{\vec{B} = \frac{\mu_0 i}{2}\,\hat{z} \times \vec{a}}$$ Uniform B-field!

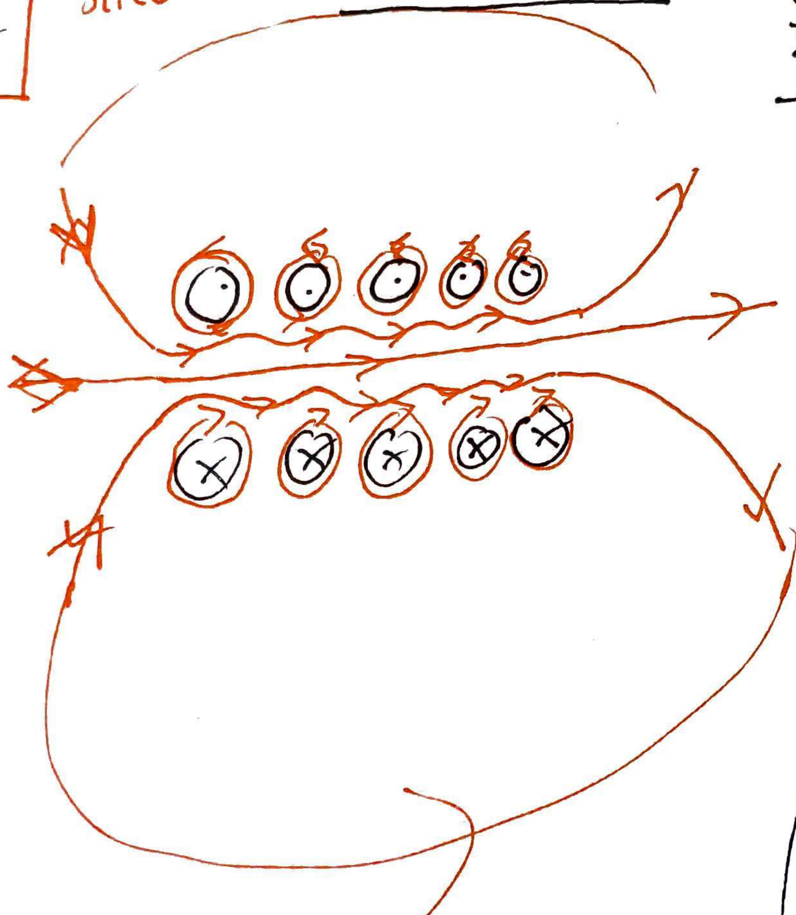


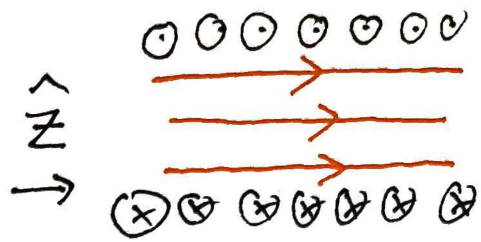
$\vec{B}$ uniform inside the cavity.

2] Solenoid – Magnetic analog to parallel plate capacitor.



Vertical slice ↪ Cross-section:



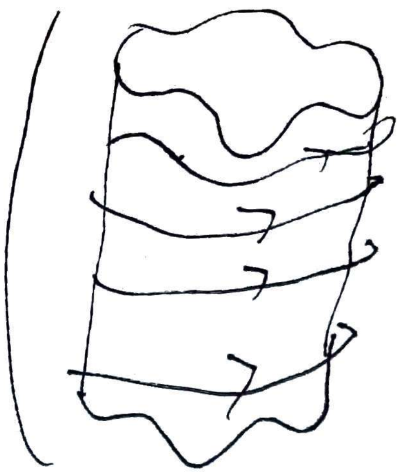


Far from ends ~~the~~ uniform $\vec{B}$ field

$\vec{B} = 0$ outside

$$\boxed{\vec{B} = B\hat{z}}$$

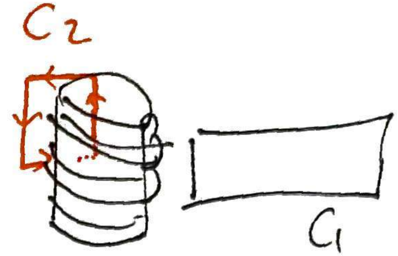
inside.

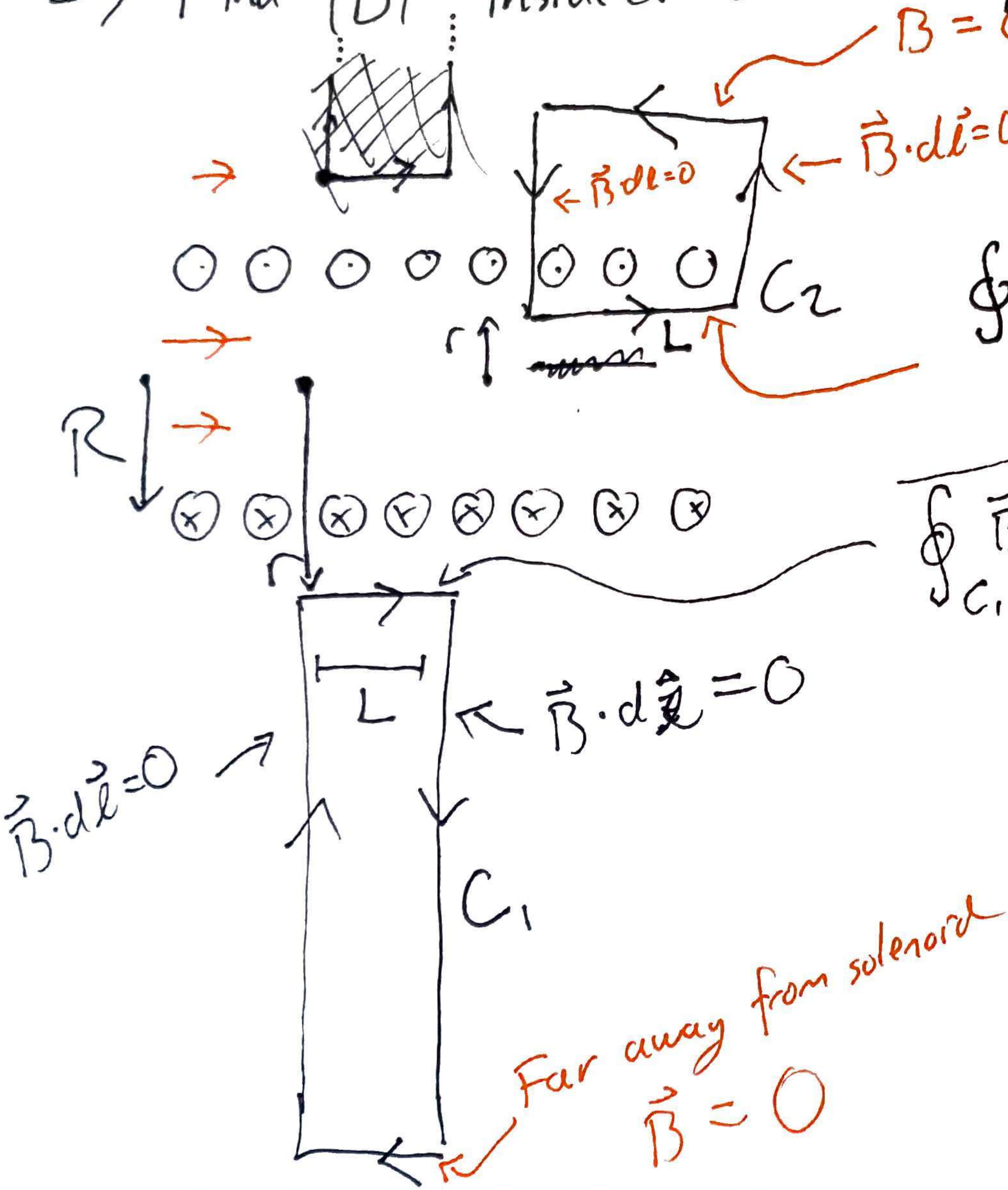


(Any shape of infinite-L solenoid has $\vec{B} = B\hat{z}$ inside, $\vec{B} = 0$ outside.)

Consider ∞-L solenoid,
assume $\vec{B}$ is in $\hat{z}$ direction everywhere.

↪ Find $|B|$ inside & outside using Ampere's law.





$$\oint_{C_2} \vec{B}\cdot d\vec{l} = B(r)\cdot L = \mu_0 I_{enc}$$

$$\boxed{B(r) = \mu_0 \frac{I_{enc}}{L} = \mu_0 I \frac{N_{wire}}{L}}$$

$$\oint_{C_1} \vec{B}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{l} + 0 + 0 + 0$$

$$= \cancel{B_{outside} L}$$

$$= B(r)\cdot L$$

$$I_{enc} = 0$$

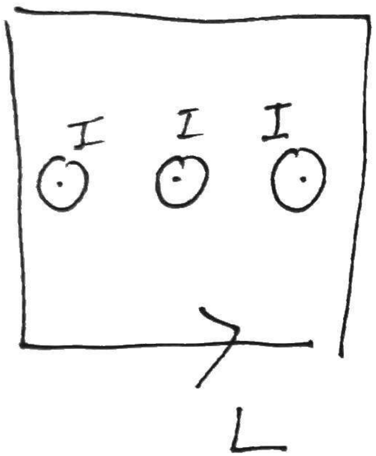
$$\Rightarrow \boxed{B(r) = 0 \quad r > R}$$

$$\oint_{C_2} \vec{B}\cdot d\vec{l} = B(r)\cdot L + 0 + 0 + 0$$
$$= B(r)\cdot L$$

$$B(r)L = \mu_0 I_{enc}$$

$$\hookrightarrow B(r) = \mu_0 \frac{I_{enc}}{L}$$

* Does not depend on r



$$B(r) = \mu_0 I \frac{N_{wires}}{L}$$

$L = 1$

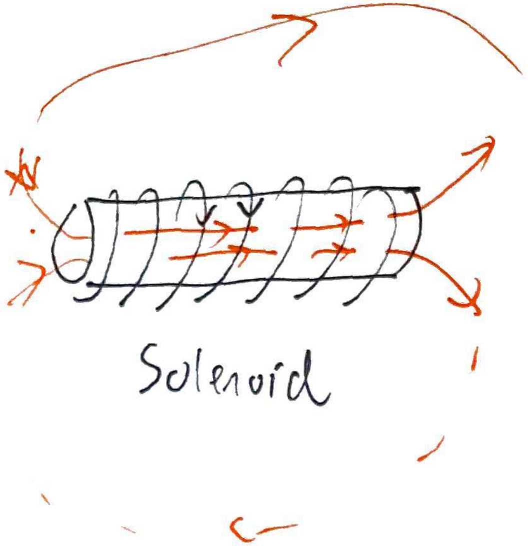
$\Rightarrow B(r) = \text{const.}$ inside.

$$= \mu_0 I n$$

$n = \frac{\# \text{ wires}}{\text{unit length}}$

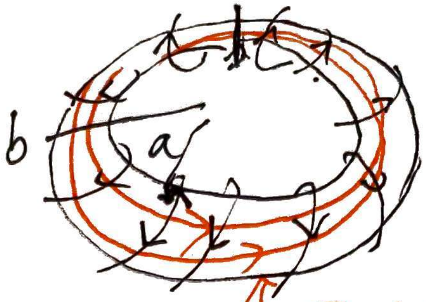
Inside: $\boxed{\vec{B} = \mu_0 I n \hat{z}}$ $r < R$

4] Toroid



Solenoid

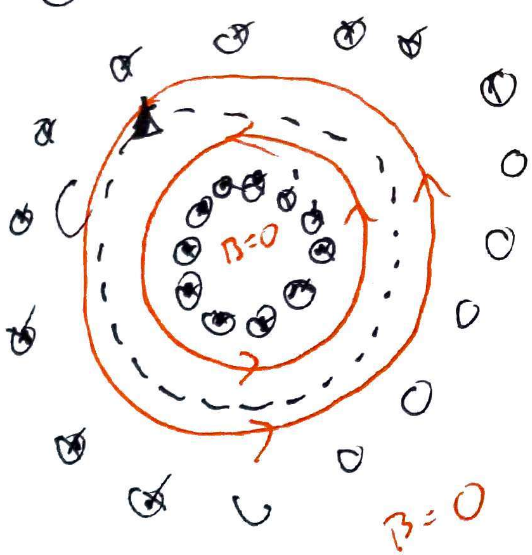
Bend into donut



B inside

$\vec{B}=0$ outside

Cross-section:



⟷ Rotational sym

$$\vec{B} = \underline{B(r)}\,\hat{\theta}$$

Use Ampere to calculate.

Ampere loop C = circle of radius r

$$\vec{B}\cdot d\vec{l} = (B(r)\,\hat{\theta})\cdot(r\,d\theta\,\hat{\theta})$$
$$= B(r)\,r\,d\theta$$

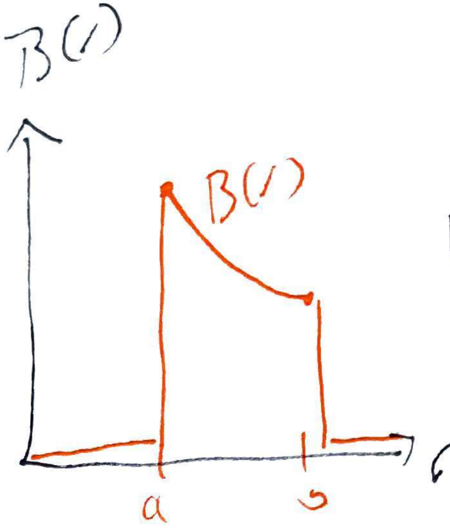
$$\oint_{C(r)} \vec{B} \cdot d\vec{\ell} = \oint_{C(r)} B(r)\, r\, d\theta$$

$$= B(r)\, r \int_0^{2\pi} d\theta = 2\pi r\, B(r)$$

$$I_{enc}(r) = \text{const.}$$

$$= I \cdot N_{loops}$$

$$\Rightarrow \quad 2\pi r\, B(r) = \mu_0 I_{enc} = \mu_0 I N_{loops}$$



$$\boxed{B(r) = \frac{\mu_0 I N_{loops}}{2\pi r}}$$

$a < r < b$

(Inside)

$$B = 0 = I_{\substack{net \\ enc.}}$$

Outside: $r < a$

$r > b$